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# MATHEMATICS magazine

# MATHEMATICS MAGAZINE

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# A NOTE ON THE SOLUTIONS OF THE EQUATION $f'(x) = f(x + a)$

John W. Green

1. Introduction. Consider the equation

$$(1) \qquad f'(x) = f(x + a),$$

where $f$ is a function of the real variable $x$, continuous for all $x$, and $a = 0$. It is obvious that any solution is of class $C_\infty$. Not much more can be said if no more restrictions are placed on $f$; it is possible to start with a more or less arbitrary function $f$ of class $C_\infty$ in an interval of length $a$, and define $f$ recursively so as to satisfy (1) for all $x$. However by placing some very light restriction on $f$ it is frequently possible to place it in some familiar function class or even to specify it exactly. For example, Herbert Robbins [2] proved that if $f$ is periodic, then it is a linear combination of $\sin x$ and $\cos x$, and $a$ is necessarily of the form $\pi(2n + \frac{1}{2})$. In the following, several other examples of this sort of result will be exhibited. Also analytic solutions of (1) will be given for any value of $a$.

2. Odd and even solutions of (1). As an example of the possibility of classifying accurately the solutions of (1) when other conditions are added, consider the case of even $f$: $f(-x) = f(x)$. $f'$ is an odd function, and we have, making alternate use of this fact and of equation (1), $f'(x) = -f'(-x) = -f(-x + a)$; $f''(x) = +f'(-x + a) = -f'(x - a) = -f(x)$. That is, $f''(x) = -f(x)$, which implies that $f = A \sin x + B \cos x$ and that $a = \pi/2$. By the evenness of $f$, we have $f = B \cos x$. Similarly if $f$ is odd, it is a constant multiple of $\sin x$.

3. Positive solutions of (1). Suppose that $a > 0$, that $f$ is a solution of (1), and that $f(x) \geqq 0$ for all $x$. Since $f^{(n)}(x) = f(x + na)$, it follows that all derivatives of $f$ are non-negative, and consequently that $f$ is what is called an absolutely monotone function. It is well known that such a function is entire, the proof being but a simple exercise in the use of Taylor's formula.

Since $f$ is entire, its Maclaurin expansion* converges for all $x$; $f(x) = \Sigma A_n x^n/n!$, where $A_n = f^{(n)}(0) = f(na)$. Since $f$ is increasing, $A_{n+1} \geqq A_n \geqq A_0$, and so for positive $x$, $f(x) \geqq A_0 x^n/n! = A_0 \; exp \; x \equiv A_0 e^x$. Thus $A_n = f(na) \geqq A_0 \; exp \; na$. Resubstituting in the Maclaurin series, we see that $f(x) \geqq \Sigma(A_0 \; exp \; na)x^n/n! = A_0 \; exp \; [x \; exp \; a]$. Repeating

* All summations go from 0 to $\infty$ unless otherwise indicated.

this process, we get $f(x) \geqq A_0 \; exp\; [x \; exp \; [a \; exp \; a]]$ and in general, $f(x) \geqq A_0 \; exp \; [x \; exp \; [a \; exp \; [. \; . \; .]]]$.

Define $S_1(a) = exp \; a$, $S_n(a) = exp \; [aS_{n-1}(a)]$. What we have shown is that $f(x) \geqq A_0 \; exp \; [x \; exp \; S_n]$. Thus $S_n$ must remain bounded for all $n$. It is a simple matter to see that this sequence is non-decreasing and hence if bounded, has a limit $S(a)$. This limit clearly must satisfy the equation $S(a) = exp \; [aS(a)]$, or $\log S/S = a$. The maximum value of $\log S/S$ for real $S$ is $1/e$, attained at $S = e$. Hence $a \leqq 1/e$. It is but an exercise in the calculus to show that if $a \leqq 1/e$, the limit approached by $S$ is the smaller of the two roots of $\log S/S = a$. What we have shown may be expressed as follows:

*If $f$ is a non-negative solution of* (1), *not identically vanishing, then $a \leqq 1/e$, and for $x \geqq x_0$, $f(x) \geqq f(x_0) \; exp \; [(x - x_0)S(a)]$, where $S$ is the smaller root of* $\log S/S = a$.

4. Exponential solutions of (1). The preceeding analysis suggests that $exp \; [xS(a)]$ might itself be a solution of (1) and this is indeed easily verified to be the case. If $e^{Sx}$ is to be a solution of (1) it is necessary and sufficient that $\log S/S = a$. For any $a \neq 0$ a few topological considerations serve to shaw that this transcendental equation has infinitely many solutions, but only for $0 < a \leqq 1/e$ are any of these solutions real. A differentiation shows that the only multiple root occurs at $a = 1/e$, $S = e$, and the multiplicity is two.

5. Entire solutions of (1) of exponential type. An entire function is of exponential type provided there exist constants $A$ and $k$ such that $|f(z)| \leqq Ae^{k|z|}$ for all $z$. A solution $f$ of (1) which satisfies $|f(x)| \leqq Ae^{k|x|}$ for real $x$ is automatically entire and of exponential type. In fact for $x > 0$, in the Taylor's formula for $f$, the remainder term is $(1/n!)|x^n f^n(\theta x)| = (1/n!)|x^n f(\theta x + na)| \leqq A \cdot \frac{|x|^n}{n!} e^{k(x + na)} \to 0$ as $n \to \infty$. So $|f(z)| = |\Sigma f^{(n)}(0)z^{(n)}/n!| = |\Sigma f(na)z^n/n!| \leqq A\Sigma|e^{kna}z^n/n!| = A \; exp \; [|z|e^{ak}]$. For these functions we prove* the following:

*If $f$ is of exponential type and satisfies* (1), *then*

$$f(z) = \sum_1^n C_n e^{S_n z}, \tag{2}$$

*where the $S_n$ are solutions of* $\log S/S = a$; *except in the case $a = 1/e$, an additional term of the form $Cze^{ez}$ may be present.*

For convenience we consider $a$ real. This is not essential and the

*The author understands from a conversation with G. W. Mackey that the latter has obtained this same result by a different method.

general case can be treated by a suitable alteration of the integration contours used. We consider the Laplace transform of $f$:

$$(3) \qquad g(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

where $s = \sigma + i\tau$. (For the facts concerning Laplace Transforms used here, see [1], pp. 61-65). The integral in (3) converges absolutely if $\sigma > k$. Let us write

$$(4) \qquad g(s) = \int_0^a e^{-st} f(t)\,dt + \int_a^\infty e^{-st} f(t)\,dt.$$

If in the second integral in the right hand member of (4), $f(t)$ is replaced by $f'(t - a)$, $t - a$ replaced by $t'$ and an integration by parts performed, there results

$$(5) \qquad g(s) = \frac{1}{e^{-as}(s - e^{as})} \left\{ \int_0^a e^{-st} f(t)\,dt - f(0)\,e^{-as} \right\}.$$

From (5) it is clear that $g$ is a meromorphic function whose poles can only be at the zeros of $s = e^{as}$. Furthermore, it is known ([1], p. 64), that the Laplace transform of any function of exponential type is analytic at infinity and vanishes there. Hence there is a circle in the $s$-plane outside of which $g$ has no poles, and so the number of poles of $g(s)$ is finite. Let $S_1$, $S_2$, ... $S_n$ be these poles. If the double root of $s = e^{as}$ is not present, then

$$(6) \qquad g(s) = \sum_{n=1}^{N} \frac{C_n}{s - S_n} + h(s),$$

where $h(s)$ is entire. But $h(s)$ vanishes at $\infty$ and must be zero. For a function $f$ of exponential type, the inversion formula for the Laplace transform is ([1], p. 65.

$$(7) \qquad f(z) = \frac{1}{2\pi i} \oint g(s)\,e^{sz}\,ds,$$

where $C$ is a circle in the $s$-plane of radius greater than $k$ with center at $s = 0$. Substituting (6) in (7),

$$(8) \qquad f(z) = \frac{1}{2\pi i} \oint \sum_1^N \frac{C_n e^{sz}}{s - S_n}\, ds = \sum_1^N C_n e^{S_n z} .$$

If $a = 1/e$, $g(s)$ in (6) may have an extra term of the form $1/(s - e)^2$. Correspondingly in $f(z)$ there may be the term $ze^{ez}$.

6. Bounded and periodic solutions of (1). A bounded solution of (1) is of exponential type and hence of the form (8). A term of (8) is of the form $e^{\alpha x}(A \cos x + B \sin x)$ for $x$ real. Clearly $\alpha = 0$ for $f$ to be bounded for real $x$. Furthermore the only pure imaginary solution of $\log s/s = a$ is $s = 1$ when $a = \pi/2 \pmod{2\pi}$. Thus we have shown

*If $f$ is bounded and satisfies* (1) *then* $f = A \sin x + B \cos x$, *and* $a = \pi/2 \pmod{2\pi}$.

A periodic solution is bounded and so the above gives another proof of Robbins' result.

[1]. G. Doetsch, Laplace Transformation, New York, 1943.

[2]. Herbert Robbins, Bull. Amer. Math. Soc. 50, pp. 750-752 (1944).

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# EVALUATION OF CERTAIN CLASSES OF INFINITE NUMERICAL SERIES IN CLOSED FORM

Bill W. Fain

Some interesting evaluations of infinite numerical series result from the product of two infinite Fourier series. Several classes of these infinite numerical series are herein evaluated in closed form[1]. The values are obtained with the aid of the rule for formally multiplying trigonometrical series, as developed by Adolf Hurwitz[2].

The rule has been expressed by E. W. Hobson[3] as follows:

"If two trigonometrical series

$$1/2a_0 + (a_1 \cos x + b_1 \sin x) + \cdots + (a_n \cos nx + b_n \sin nx) + \cdots \quad (1)$$

$$1/2\alpha_0 + (\alpha_1 \cos x + \beta_1 \sin x) + \cdots + (\alpha_n \cos nx + \beta_n \sin nx) + \cdots \quad (2)$$

be multiplied together, as if the series are finite, and the result be arranged as a trigonometrical series, we obtain the series

$$1/2A_0 + (A_1 \cos x + B_1 \sin x) + \cdots + (A_n \cos nx + B_n \sin nx) + \cdots \quad (3)$$

where

$$1/2A_0 = 1/4a_0\alpha_0 + 1/2 \sum_{n=1} (a_n\alpha_n + b_n\beta_n) \quad (4)$$

$$A_n = 1/2a_0\alpha_n + 1/2 \sum_{p=1} [a_p(\alpha_{p+n} + \alpha_{p-n}) + b_p(\beta_{p+n} + \beta_{p-n})] \quad (5)$$

$$B_n = 1/2a_0\beta_n + 1/2 \sum_{p=1} [a_p(\beta_{p+n} - \beta_{p-n}) - b_p(\alpha_{p+n} - \alpha_{p-n})] \quad (6)$$

where it is assumed that $\alpha_{-k} = \alpha_k$, $\beta_{-k} = -\beta_k$.

In this expression, the numbers $a_0$, $a_n$, $b_n$ and $\alpha_0$, $\alpha_n$, $\beta_n$ may be interchanged.

The series (3) is said to be the formal product of the series (1) and (2).

In case the series $\sum_{n=1} a_n$, $\sum_{n=1} b_n$, $\sum_{n=1} \alpha_n$, $\sum_{n=1} \beta_n$ are all absolutely convergent, the series (1), (2) converge absolutely and uniformly to continuous sum-functions $f_1(x)$, $f_2(x)$. In that case the Cauchy-multiplication of the series (1) and (2) yields an absolutely and uniformly convergent series of which the sum-function is the product $f_1(x)f_2(x)$. The series may then be arranged in the form (3) without altering the character of its convergence; and therefore the series (3) converges to $f_1(x)f_2(x)$.

Next, let (1) and (2) be the Fourier's series which correspond to two summable functions $f(x)$, $g(x)$.

If either (1), $\{f(x)\}^2$, $\{g(x)\}^2$ are both summable in the interval $(-\pi, \pi)$, or (2), one of the functions $f(x)$, $g(x)$ is summable, and the other of bounded variation, in the interval $(-\pi, \pi)$, the formal product, of which the coefficients are given by (4), (5) and (6), is the Fourier's series corresponding to the product $f(x)g(x)$."

Equations (3), (4), (5) and (6) show that the product of two infinite Fourier series is another Fourier series with some of the coefficients containing infinite numerical series. If these Fourier series represent functions that are analytical, it is possible to evaluate the infinite numerical series in closed form.

### Illustration

The Fourier equation of $y = x$ $(0 \leqq x \leqq 2\pi)$, the saw toothed curve, is

$$y = x = \pi - 2\left[\sin x + 1/2 \sin 2x + 1/3 \sin 3x + \cdots\right] \tag{7}$$

If Eq. (7) is multiplied by itself, the resultant equation,

$$\begin{aligned} y = \frac{4\pi^2}{3} &+ 4\Big[\Big(\Big\{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots\Big\}\Big) \cos x \\ &+ \Big(\Big\{\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots\Big\}\Big) \cos 2x \\ &+ \Big(\Big\{\frac{1}{1\cdot 4} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 6} + \cdots\Big\} - 1/2\Big) \cos 3x \\ &+ \Big(\Big\{\frac{1}{1\cdot 5} + \frac{1}{2\cdot 6} + \frac{1}{3\cdot 7} + \cdots\Big\} - 11/24\Big) \cos 4x \\ &+ \cdots\cdots\Big] \\ &- 4\pi\left[\sin x + 1/2 \sin 2x + \cdots\cdots\right] \end{aligned} \tag{8}$$

is the Fourier equation of the curve $y = x^2$ $(0 < x < 2\pi)$. Generalizing Eq. (8)

$$y = \frac{4\pi^2}{3} + 4 \sum_{k=1}^{\infty} \left\{\left[\frac{1}{1\cdot(1+k)} + \frac{1}{2\cdot(2+k)} + \frac{1}{3\cdot(3+k)} + \cdots\right] - \tfrac{1}{2} \sum_{n=1}^{k-1} \left(\frac{1}{n} \cdot \frac{1}{k-n}\right)\right\} \cos kx - \sum_{k=1}^{\infty} \frac{4\pi}{k} \sin kx \tag{9}$$

Since the Fourier series for $y = x^2$ $(0 < x < 2\pi)$ can be obtained independently by conventional Fourier methods, it is possible to evaluate the infinite numerical series that are involved in the coefficients of the cosine terms. The equation obtained by conventional methods is

$$y = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2} \cos kx - \sum_{k=1}^{\infty} \frac{4\pi}{k} \sin kx \tag{10}$$

Equating coefficients of the cosine terms in Eq. (9) and Eq. (10) gives

$$\frac{1}{1\cdot(1+k)} + \frac{1}{2\cdot(2+k)} + \frac{1}{3\cdot(3+k)} + \cdots = \frac{1}{k^2} + \frac{1}{2}\sum_{n=1}^{k-1}\left[\frac{1}{n}\cdot\frac{1}{k-n}\right] \tag{11}$$

That is, the value (on the right of Eq. (11)) of the class of infinite numerical series (on the left) was obtained by equating the coefficients in the Fourier series for $y = x^2$, which resulted from multiplying the Fourier series for $y = x$ by itself, to the coefficients in the Fourier series for $y = x^2$ obtained in the conventional manner. Therefore, since the summation on the right is *finite*, the infinite series on the left, which will be called for convenience Class I, can be evaluated in closed form for any integral value of $k$ greater than zero.

Evaluation of Specific Series, Class I

$$k = 1: \quad \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \cdots = 1 \tag{12}$$

$$k = 2: \quad \frac{1}{1\cdot3} + \frac{1}{2\cdot4} + \frac{1}{3\cdot5} + \frac{1}{4\cdot6} + \cdots = \frac{3}{4} \tag{13}$$

$$k = 3: \quad \frac{1}{1\cdot4} + \frac{1}{2\cdot5} + \frac{1}{3\cdot6} + \frac{1}{4\cdot7} + \cdots = \frac{11}{18} \tag{14}$$

$$k = 4: \quad \frac{1}{1\cdot5} + \frac{1}{2\cdot6} + \frac{1}{3\cdot7} + \frac{1}{4\cdot8} + \cdots = \frac{25}{48} \tag{15}$$

$$k = 5: \quad \frac{1}{1\cdot6} + \frac{1}{2\cdot7} + \frac{1}{3\cdot8} + \frac{1}{4\cdot9} + \cdots = \frac{137}{300} \tag{16}$$

$$k = 6: \quad \frac{1}{1\cdot7} + \frac{1}{2\cdot8} + \frac{1}{3\cdot9} + \frac{1}{4\cdot10} + \cdots = \frac{157}{360} \tag{17}$$

$$k = 7: \quad \frac{1}{1\cdot8} + \frac{1}{2\cdot9} + \frac{1}{3\cdot10} + \frac{1}{4\cdot11} + \cdots = \frac{1089}{2940} \tag{18}$$

In a similar manner the evaluation for another class of series, Class II, which is

$$\frac{1}{1^2\cdot(k+1)^2}+\frac{1}{2^2\cdot(k+2)^2}+\frac{1}{3^2\cdot(k+3)^2}+\cdots \tag{19}$$

can be made by first multiplying the Fourier series of $y = x^2$ $(-\pi < x < \pi)$ by itself, and then comparing the product with the Fourier series for $y = x^4$. The evaluation for the series is

$$\frac{1}{1^2\cdot(k+1)^2}+\frac{1}{2^2\cdot(k+2)^2}+\cdots=\frac{k^2\pi^2-9}{3k^4}-\frac{1}{2}\sum_{n=1}^{k-1}\frac{1}{n^2(k-n)^2} \tag{20}$$

That is, the value (on the right of Eq. (20)) of the class of infinite numerical series (on the left) was obtained by equating the coefficients in the Fourier series for $y = x^4$, which resulted from multiplying the Fourier series for $y = x^2$ by itself, to the coefficients of the Fourier series for $y = x^4$ obtained in the conventional manner. The series on the right of Eq. (20) is *finite* and, therefore, the infinite series is evaluated in closed form.

Evaluation of Specific Series, Class II

$$k=1:\ \frac{1}{1\cdot 2^2}+\frac{1}{2^2\cdot 3^2}+\frac{1}{3^2\cdot 4^2}+\cdots=\frac{\pi^2-9}{3} \tag{21}$$

$$k=2:\ \frac{1}{1\cdot 3^2}+\frac{1}{2^2\cdot 4^2}+\frac{1}{3^2\cdot 5^2}+\cdots=\frac{4\pi^2-33}{48} \tag{22}$$

$$k=3:\ \frac{1}{1\cdot 4^2}+\frac{1}{2^2\cdot 5^2}+\frac{1}{3^2\cdot 6^2}+\cdots=\frac{4\pi^2-31}{108} \tag{23}$$

$$k=4:\ \frac{1}{1\cdot 5^2}+\frac{1}{2^2\cdot 6^2}+\frac{1}{3^2\cdot 7^2}+\cdots=\frac{48\pi^2-355}{2304} \tag{24}$$

$$k=5:\ \frac{1}{1\cdot 6^2}+\frac{1}{2^2\cdot 7^2}+\frac{1}{3^2\cdot 8^2}+\cdots=\frac{1200\pi^2-8557}{90{,}000} \tag{25}$$

$$k=6:\ \frac{1}{1\cdot 7^2}+\frac{1}{2^2\cdot 8^2}+\frac{1}{3^2\cdot 9^2}+\cdots=\frac{1200\pi^2-8309}{129{,}600} \tag{26}$$

$$k = 7: \frac{1}{1\cdot 8^2} + \frac{1}{2^2\cdot 9^2} + \frac{1}{3^2\cdot 10^2} + \cdots = \frac{705,600\pi^2 - 4,768,337}{103,723,200} \qquad (27)$$

The generalized expression for the value of a series of the form, designated as Class III,

$$\frac{1}{1\cdot(k+1)} + \frac{1}{3\cdot(k+3)} + \frac{1}{5\cdot(k+5)} + \cdots \qquad (28)$$

where $k$ is an *even* integer greater than zero, is found by first multiplying the Fourier series for $y = x$, $(0 < x < 2\pi)$ by the series for $y = 1$ $(0 < x < \pi)$, $y = 0$ $(\pi < x < 2\pi)$, and then comparing coefficients with the Fourier series for $y = x$ $(0 < x < \pi)$, $y = 0$ $(\pi < x < 2\pi)$, obtained in the conventional manner. Consequently, the series

$$\frac{1}{1\cdot(k+1)} + \frac{1}{3\cdot(k+3)} + \cdots = \frac{1}{4}\sum_{n=1}^{k-1}\frac{1}{n}\left[\frac{(-1)^{n+1}+1}{k-n}\right] \qquad (29)$$

Evaluation of Specific Series, Class III

$$k = 2: \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2} \qquad (30)$$

$$k = 4: \frac{1}{1\cdot 5} + \frac{1}{3\cdot 7} + \frac{1}{5\cdot 9} + \cdots = \frac{1}{3} \qquad (31)$$

$$k = 6: \frac{1}{1\cdot 7} + \frac{1}{3\cdot 9} + \frac{1}{5\cdot 11} + \cdots = \frac{23}{90} \qquad (32)$$

$$k = 8: \frac{1}{1\cdot 9} + \frac{1}{3\cdot 11} + \frac{1}{5\cdot 13} + \cdots = \frac{22}{105} \qquad (33)$$

By subtracting the series of Class II from the series of Class I, it is possible to evaluate the series, Class IV,

$$\frac{1}{2\cdot(2+k)} + \frac{1}{4\cdot(4+k)} + \frac{1}{6\cdot(6+k)} + \cdots \qquad (34)$$

where $k$ is positive and even.

Evaluation of Specific Series, Class IV

$$k = 2: \frac{1}{2\cdot 4} + \frac{1}{4\cdot 6} + \frac{1}{6\cdot 8} + \cdots = \frac{1}{4} \qquad (35)$$

$$k = 4: \quad \frac{1}{2\cdot 6} + \frac{1}{4\cdot 8} + \frac{1}{6\cdot 10} + \cdots\cdot = \frac{9}{48} \tag{36}$$

$$k = 6: \quad \frac{1}{2\cdot 8} + \frac{1}{4\cdot 10} + \frac{1}{6\cdot 12} + \cdots\cdot = \frac{65}{360} \tag{37}$$

The usefulness of this method is that an *entire class* of infinite numerical series can be evaluated in closed form. Evidently, other classes of series can be evaluated in closed form by the same technique as herein demonstrated.

---

[1] Knopp, Konrad. *Theory and Applications of Infinite Series.* London, Blackie and Son, 1928, p. 230.

[2] Hurwitz, Adolf. "Sur quelques applications geometrique des series de Fourier," *Annales de l'Ecole Normale Superieure,* 3 me serie, Vol. 19.

[3] Hobson, E. W. *The Theory of Functions of a Real Variable* Washington, D. C., Harren Press, 1950. vol. II, pp. 585—586.

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# THE PERSONAL SIDE OF MATHEMATICS

Articles intended for this Department should be sent to the Mathematics Magazine, 14068 Van Nuys Blvd., Pacoima, California.

## MATHEMATICS AND MATHEMATICIANS FROM ABEL TO ZERMELO*

Einar Hille

From the historical point of view, mathematics may be considered as a series of events in space-time associated with definite persons or, if you prefer, as a type of finite geometry where the coordinates measure, for instance, person, place, time, quality, and field of research. We should be able to get some idea of the structure of this geometry by considering suitable chosen cross sections. Two such cross sections will be used below.

In the first study the time coordinate is held approximately constant at $t$ = 1852 A.D. and we shall list the prominent mathematicians of a century ago together with brief accounts of their work and background. In the second study, the field of research is held approximately constant and the time coordinate is restricted. Here we shall follow the development of analysis, in particular, that of complex function theory, over a period of a century, roughly from 1820 to the time of the first world war.

### I. Who Was Who in Mathematics in 1852?

1. *France*. We shall try to reconstruct a Who's Who in Mathematics around 1850. We have to keep in mind that there were much fewer mathematicians in those days, fewer periodicals, little personal contact, no mathematical societies, no scientific meetings or congresses. Mathematics flourished in France, Germany, and Great Britain; outside of these countries research mathematicians were few and far between. There was interchange of ideas, however, mathematicians did write to each other, young students traveled abroad to study and so forth.

In this survey it is reasonable to start with France where there is a splendid mathematical tradition going back to the seventeenth

* This paper is based upon two talks presented to the Mathematics Colloquium of Yale University in May 1952. It is hoped that the contents might prove to be of interest to a wider public. The material is culled from F. Cajori's and G. Loria's histories of mathematics and F. Klein, Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert.

century. The Revolution, having abolished the old Académie des Sciences, founded in 1666, and having liquidated most of its members, found it necessary for military reasons to revive the Academy as the Institut National in 1795 and to found the Ecole Normale Supérieure and the École Polytechnique. The latter became the cradle of French mathematics, a role that it kept for about a century until this function was taken over by the École Normale. Organized and led by the geometer Gaspard Monge (1746 – 1818), the École Polytechnique gave an intense two year course concentrating on mathematics, descriptive geometry and drawing. Entrance was severely restricted, only 150 students could be accepted each year on the basis of searching examinations. Galois failed twice in this examination, during the first century of its existence J. Hadamard held the highest score for admission with 1875 points out of a possible 2000. Only the best brains could secure admission, only hard work could carry the student through the intense training and the gruelling final examinations. Most French mathematicians of the nineteenth century belonged to this school in one capacity or another as student, instructor, professor or examiner.

The time of the revolution and the first empire saw the flourishing of mathematics in France. J. L. Lagrange (1736-1813) spent the last twenty years of his wandering life in France, men like J. Fourier (1768-1830), P. S. Laplace (1749-1827), A. M. Legendre (1752-1833), and S. D. Poisson (1781-1840) brought French mathematics to new heights. A reaction is noticeable around 1830 and lasted for almost fifty years. The old heroes were dying off and the replacements are perhaps a shade less impressive.

The big name in Paris in 1850 was Augustin-Louis Cauchy (1789-1857). As an ardent royalist Cauchy had given up his professorship at the École Polytechnique after the July revolution in 1830 and had gone in exile with the Bourbons; they had trouble with loyalty oaths in those days too. He returned to Paris in 1838 and taught at a Jesuit college. In 1848 the February revolution opened the way to a professorship at the Sorbonne without any oath and Napoleon III did not interfere with him when the constitution was changed again in 1852. Cauchy was above all an analyst, the first rigorous analyst, the founder of function theory, of the theory of ordinary and partial differential equations, who also made important contributions to elasticity and optics as well as to algebra. We shall return to the work of Cauchy in part II of this paper.

At the Collège de France in Paris we would have found Joseph Liouville (1809-1882), the man who started the mathematical theory of boundary value problems for linear second order differential equations, who produced the first integral equation and the first resolvent, but also the founder of the theory of transcendental numbers. He was the editor of the Journal de Mathématiques Pures et Appliquées, started in 1835 and still known as the Journal de Liouville. In connection with Liouville we often think of Jacques-Charles-Francois Sturm (1805-1855); he was Poisson's successor as professor of mechanics at the Sorbonne in 1850. Lesser lights, but famous in their days, were August-Albert Briot (1817-1882) and Jean-Claude Bouquet (1819-1885), great friends who collaborated in the theory of differential equations and elliptic functions. A young man by the name of Charles Hermite (1822-1901) was struggling for recognition, he still had to wait until 1869 for a professorship. Those mentioned so far were all analysts, but there were also geometers around: Jean-Victor Poncelet (1788-1867), the father of projective geometry, Charles Dupin (1784-1873) and Michel Chasles (1793-1880), were active. The algebraic geometer Chasles, who created the enumerative methods of geometry, devoted almost twenty-five years of his life to banking in his home town Chartres; in 1846 the Sorbonne created a professorship in higher geometry for him. The private fortune that he had amassed before returning to mathematics led him into collecting autographs and he was swindled into buying numerous forgeries. His most famous acquisition was a letter supposedly written by Mary Magdalene from Marseille to Saint Peter in Rome! Even Chasles ultimately had to admit that he had been swindled. Among French mathematicians we find few algebraists, the analysts attended to the theory of equations, and group theory was in its infancy. The publication of the work of Évariste Galois (1811-1832) did not take place until 1846.

A characteristic feature of French mathematics, then as well as now, is the strong centralization. Paris, that is, the Sorbonne, Collège de France, École Polytechnique, École Normale Supérieure, and the many technical schools, has most of the desirable positions while the provincial universities vary in importance, but cannot hold their own against Paris. There were numerous facilities for publication in France. Joseph-Diaz Gergonne (1771-1859), who competed with Poncelet in discovering the principle of duality, published the first mathematical journal in the world, the Annales des mathématiques pures et appliquées, in Nîmes during 1810-31 where he was then professor at the local lycée. Liouville started his journal in 1835. Notes could be published quickly in the Comptes Rendus de l'Académie des Sciences at Paris which has appeared weekly since August 1835. The restriction to short notes (originally four pages, now two) was due to Cauchy flooding the Academy with his publications. The École Polytechnique and later also the École Normale had their own journals.

2. *Germany*. Turning now to Germany we find a totally different picture. German mathematics is unimportant during the eighteenth century and is essentially a one-man-show during the first quarter of the nineteenth. From then on German mathematics is in a steady upward march until the time of the first world war and even during the empire this movement is not centralized as in France but is attached to a number of local centers. The Germany of 1850 was a geographical and cultural

unit but not a political one. It was split into a large number of autonomous kingdoms, grandduchies, duchies, principalities, free cities and what have you, and this caleidoscopic picture reflected itself in a large number of independent and thriving universities. Apparently the many political boundaries interfered very little with the students and the professors who moved freely around. Prussia had the largest number of universities of all German lands: Berlin was the largest and was outstanding in mathematics but there were also important schools in Bonn and Königsberg. In the kingdom of Hanover we find the Georgia Augusta University at Göttingen. Incidentally, it was founded two hundred years ago by the then elector of Hanover who as King George II of Great Britain and Ireland advanced the cause of learning in the then colonies by the founding of the College of New Jersey (Princeton), King's College (Columbia), and Queen's College (Rutgers). Leipzig was the intellectual center of Saxony, in Bavaria Erlangen, in Baden Freiburg and Heidelberg, and in the Thuringian maze of principalities Jena kept the torch burning.

The eighteenth century was the time when the Swiss Leonard Euler and the Frenchman Lagrange, born in Italy, lived and worked in Berlin and filled the memoirs of the Prussian Academy, but of native German mathematicians there were very few. The nineteenth century starts out with Carl Friedrich Gauss (1777-1855), a giant in mathematics, astronomy, geodesy and magnetism, who got his doctor's degree in Helmstadt in 1799 and from 1807 on spent his life in Göttingen as director of the astronomical observatory. His publications in number theory, algebra, and differential geometry were basic, but many later discoveries in function theory and geometry had been anticipated by Gauss who published only sparingly.

Though Gauss had only few direct pupils, he had started something in Germany which was of lasting importance for mathematics. This development gathered momentum in the eighteen twenties and led to a flourishing of German mathematics which lasted until the days of Hitler. For the next phase of the development we have to turn to Berlin and Königsberg. An outward sign of the changing times was the founding in Berlin in 1826 of the Journal für die reine und angewandte Mathematik by August Leopold Crelle (1780-1855), usually known as Crelle's Journal (1855-1880 Borchardt's Journal after the then editor). Crelle started out with a scoop: five memoirs by the young Norwegian mathematician Niels Henrik Abel (1802-1829) but there were also papers by Carl Gustav Jacob Jacobi (1804-1851) and by Jacob Steiner (1796-1863). In volume 3 appear the names of Peter Gustav Lejeune Dirichlet (1805-1859), August Ferdinand Möbius (1790-1868), and Julius Plücker (1801-1868). Crelle's Journal soon became known as the Journal für die reine, unangewandte Mathematik, much to the embarassment of its editor who was much more prominent as an engineer than as a mathematician. In addition to Crelle, the German mathematicians had at their disposal a large number of academy publications. The Mathematische Annalen did not start until 1868.

Jacobi got his degree in Berlin but came to Königsberg in 1826 where he did his fundamental work on elliptic functions in competition with Abel (see further in Part II). He became ordinary professor in 1831, but retired owing to ill health in 1842, whereupon he moved back to Berlin where he got a research chair without teaching duties. Jacobi is the founder of the Königsberg school to which belonged men like L. O. Hesse (1811-1874) and R. F. A. Clebsch (1833-1872), followed in later years by A. Hurwitz (1859-1919), D. Hilbert (1862-1942) and H. Minkowski (1864-1909). In more recent years men like K. Knopp (1882- ) and G. Szegö (1895- ) held professorships there. The last mathematician of note to get his training in Königsberg seems to have been Th. Kaluza now in Göttingen. Perhaps there will also be a Kaliningrad mathematical school. Who knows?

Dirichlet was born in the Rhineland as the son of French emigrants. He spent the years 1822-27 as teacher in a private family in Paris where he came under the influence of the great French analysts of the period. He also read and reread Gauss's Disquisitiones Arithmeticae. Both facts are strongly reflected in his mathematical life. After a short time in Breslau he came to Berlin in 1829 as Privatdozent and became ordinary professor ten years later, the first of the great masters of the Berlin school. It is typical for German conditions, however, that he accepted a call to Göttingen to become Gauss's successor in 1855. He died four years later. Dirichlet worked in number theory, in particular analytic number theory (Dirichlet's series commemorates this fact), functions of a real variable, Fourier series, potential theory etc. Dirichlet was an excellent expositor and exercized a strong influence on the younger men who came to Berlin such as Leopold Kronecker (1823-1891), F. G. Eisenstein (1823-1852), Richard Dedekind (1831-1916), and Bernhard Riemann (1821-1866). At Berlin we also find the brilliant Swiss born geometer Jacob Steiner who, however, never got beyond the extraordinary professorship to which he was appointed in 1834. Kronecker, a man of wealth, came to Berlin in 1855 and became attached to the University in 1861. In 1856 Ernst Eduard Kummer (1810-1893) was called to Berlin as the successor of Dirichlet. He was the father of ideal theory and also a geometer of note. The same year Karl Weierstrass (1815-1897) was called to Berlin: he had twelve hours a week at the Gewerbeakademie (a college for trade and commerce) and an extraordinariat at the university. The ordinary professorship did not come until 1864. The reader is referred to Part II for more details about Weierstrass.

Let us now turn to Göttingen and the year 1850. Gauss is still active. A young man by name of Riemann has just returned from a three year stay in Berlin and is now working on a dissertation which gave him the doctor's degree in 1851 and ultimate fame and glory. At this stage Riemann is scarcely influenced by Gauss, but he worked with

Wilhelm Weber, a famous physicist and one of the discoverers of the electric telegraph. From Dirichlet and Weber he picked up an interest in mathematical physics which influenced both his research and his lectures, later published in book form as Differentialgleichungen der Physik, first edited by Hattendorf, from the fourth edition by Heinrich Weber, and the seventh edition of 1925-26 by Philipp Frank and Richard v. Mises. Riemann qualified for the venia legendi in 1854, that is, for the right to give lectures as a Privatdozent. This required writing another dissertation (Habilitationsschrift) and giving a lecture (Habilitationsvortrag) before the faculty on one out of three preassigned topics. The former was his paper on trigonometric series, the latter dealt with the basic hypotheses of geometry. Riemann became ordinary professor in 1859. With Gauss, Dirichlet and Riemann as fathers of the Göttingen mathematical school a star was born which has never set.

There are few names to be added from the other German universities. Möbius (with the strip!) was professor of astronomy in Leipzig. He was a pupil of Gauss who turned him into an astronomer, but most of his production was in mathematics: number theory, combinatorics, and the barycentric calculus, his magnum opus of 1827, while the non-orientable manifolds was work done in his old age appearing in 1858. In Bonn we find another famous geometer, Julius Plücker, professor of physics and mathematics from 1838 until his death in 1868. As a physicist he was an experimental one and did pioneering work in spectral analysis which took up most of his time. But his contributions to geometry such as the Plücker coordinates in line geometry, a subject created by him, and the Plücker formulas in algebraic geometry, are fundamental. Plücker had several famous pupils, the physicist Hittorf was outstanding in spectral analysis, the mathematician Felix Klein (1849-1925) published his collected works. Finally we have to mention Christian v. Staudt (1798-1867), professor in Erlangen since 1835 until his death in 1867. Among his main achievements are his making projective geometry independent of metric considerations and the introduction of imaginary elements in geometry. His ideas ripened slowly, his Geometrie der Lage appeared in 1847, followed by three further Beiträge in 1856, 1857 and 1860.

3. *Great Britain and Ireland.* Due to the priority fights between Newton's pupils and those of Leibnitz, mathematics in the British isles had enjoyed a splendid but unhealthy isolation for over a century, the after-effects of which were not overcome until after 1900. One of the consequences of this was an almost complete lack of feeling for analysis per se. What there was of analysis went into mathematical physics. As an example we could take George Green (1793-1841) who applied analysis to electricity and magnetism. Drink carried him off to an early death, but his name lives (Green's theorem, Green's function etc.). George Gabriel Stokes (1819-1903), another Cambridge don, made fundamental contributions to optics and hydrodynamics. There is a

theorem of Stokes in the Calculus, there is also a phenomenon of Stokes in the theory of differential equations. To the same tradition belongs the work of the famous physicists James Clerk Maxwell (1831-1879) and William Thomson, Lord Kelvin (1824-1907). Sir William Rowan Hamilton (1805-1865) was professor of astronomy at Trinity College in Dublin, but is more famous for his work in optics and mechanics (the Hamiltonian equations) and as creator of vector analysis and the theory of quaternions.

But there is also another direction in British mathematics which breaks through around 1850, represented by the three names Arthur Cayley (1821-1895), George Salmon (1819-1904), and James Joseph Sylvester (1814-1897). They were personal friends. Cayley and Sylvester read for the bar together in London. Cayley and Salmon had a joint paper on the twenty-seven lines on the cubic surface in 1849 and Cayley helped Salmon to revise his Higher Plane Curves. Salmon stayed all his life at Trinity, he was a fellow from 1840 on, became professor of divinity in 1866, and provost in 1888. He is best known for his books on geometry and algebra which were much read and translated into French and German, but he also wrote books on theology.

Cayley was one of the most prolific mathematicians of all time: starting production in 1841 he produced a total of 887 papers. And this intense activity was kept up while he was reading for the bar and during the fourteen years he practiced law – he was a conveyancer – as a matter of fact, most of his basic ideas stem from this period. In 1863 he retired from practice and accepted the Sadlerian professorship of pure mathematics in Cambridge which he held until his death. Cayley's nine memoirs on quantics (= forms) are famous, he started the theory of matrices and, together with Sylvester, the theory of invariants. The very name of invariant is due to Sylvester who also introduced covariant, contragrediant, discriminant, etc.. Sylvester used to refer to himself as the new Adam since he had named so many things. Sylvester held a multitude of positions, including a brief stay at the University of Virginia in the early forties. He had been an actuary, he was called to the bar in 1850, from 1855 to 1870 he taught at the Military Academy at Woolwich, 1876-1883 he spent at The Johns Hopkins University where he started the American Journal of Mathematics as well as the high traditions of Johns Hopkins in mathematics. At seventy he accepted a call to Oxford as successor of the number theory man H.J.S. Smith (1826-1883) as Savilian professor of geometry.

If to this list we add the logicians Augustus de Morgan (1806-1871) and George Boole (1815-1864) we have a fairly good picture of British mathematics at the middle of the last century. The Cambridge Journal of Mathematics started in 1837, after four volumes it became the Cambridge and Dublin Mathematical Journal, later (in 1855) the Quarterly Journal of Pure and Applied Mathematics. The London Mathematical Society, the first of its kind, was organized in 1865.

4. *Other countries*. The compiler of our Who's Who would have found

a single research mathematician in this country, namely Benjamin Peirce (1809-1880), professor at Harvard since 1833, whose work on linear associative algebra was really fundamental. In the seventies things are beginning to stir elsewhere also; we have already referred to Sylvester's stay at Johns Hopkins. In 1871 Josiah Willard Gibbs (1839-1903) became professor of mathematical physics at Yale; his main work on thermodynamics, equilibra and the phase rule date from 1873-78. His work on vector analysis was of some importance and one or two of his students made mathematical history. The real awakening did not come until the eighteen nineties, however.

Back in Europe again, let us look briefly at the situation in Italy where the mathematical* tradition goes back to the year 1200. The beginning of the nineteenth century is meagre, however. After the death of Paolo Ruffini (1765-1822), there is not much mathematics in Italy, but the awakening in mathematics came with the general national awakening and shortly before the unification. As fathers of modern Italian mathematics it is customary to acknowledge Francesco Brioschi (1824-1897), Enrico Betti (1823-1892), Felice Casorati (1835-1890) and Luigi Cremona (1830-1903). Brioschi became professor of applied mathematics in Pavia in 1852; in 1858 he made a journey to France and Germany together with Betti and Casorati and it is from this journey that the revival is dated. Brioschi organized the technological institute of Milan from 1862 on, Cremona played a similar part in Rome, and Betti became the director of the Scuola Normale Superiore of Pisa. To these names should be added that of Eugenio Beltrami (1835-1900). Most of the Italian mathematicians turned towards algebraic geometry.

Euler spent over twenty years in Russia but, beyond filling the memoirs of the Saint Petersburg Academy (for which purpose he had been imported), there is little trace of his activities. Around 1850 we would have found at least three prominent mathematicians in Russia: Victor Jacovlevich Bouniakovski (1804-1889), Pafnuti Livovich Chebicheff (1821-1894), both in St. Petersburg (= Petrograd = Leningrad), and Nicolay Ivanovich Lobatchevsky (1773-1856) in Kazan. Bouniakovsky's inequality was the customary name in Russia for what we know as Schwarz's inequality (usually misspelt Schwartz in this country). Chebicheff started the investigation of extremal problems in function theory and problems of best approximation. He also worked with prime numbers. Lobatchevsky had been pondering over the parallel postulate since 1815; around 1826 he arrived at the alternate postulate of two parallels and his investigations were published in seven memoirs 1829-1856.

Turning to Hungary we find the other discoverer of non-euclidean geometry Janos (= John or Johann) Bolyai (1802-1860) whose discovery was published in an appendix to a mathematical memoir of his father's, Farkas (= Wolfgang) Bolyai (1775-1856), in 1832-1835. It is well known that Gauss had been thinking along similar lines; it is perhaps more than a coincidence that both Wolfgang Bolyai and Lobatchevsky's teacher Bartels in Kazan were personal friends of Gauss.

There is little activity elsewhere in Europe. Switzerland had over-exerted itself during the eighteenth century with all the Bernoullis and with Euler and was recovering from the strain. Scandinavia had produced Abel and new talent was in the making both in Norway and in Sweden, but there is nothing to report until after 1860 to 1870.

(To be continued.)

## II. *The development of analysis, primarily complex function theory, until the time of the first world war.*

5. *Cauchy*. The outstanding mathematical creation of the nineteenth century is hard to single out and the choice will largely depend upon individual preference: projective geometry, invariant theory, group theory, theory of sets, complex function theory, each of these will have its spokesmen, and the list is not exhausted. My vote is cast for complex function theory, the development of which I propose to describe on the following pages.

We have already mentioned Augustin-Louis Cauchy (1789-1857) in Part I of this paper. Cauchy laid the foundations of complex function theory in his Cours d'Analyse of 1821 where he developed the theory of the elementary functions in the complex plane. This was followed by a small memoir on complex integration in 1825, but the general integral theorem is considerably later, about 1840. That holomorphic functions can be expanded in power series he proved in 1831, while in exile in Turin. In 1843 P. A. Laurent (1813-1854) proved the expansion theorem for the annulus and in 1850 V. A. Puiseux (1820-1883) obtained the expansion in fractional powers valid at algebraic branch points.

6. *Abel and Jacobi*. Parallel with this general development and at the time overshadowing it, there was a special trend dealing with elliptic and Abelian functions. To get the background here we have to go back to the latter half of the eighteenth century. Mathematicians like Euler, Lagrange and Legendre were then struggling with integrals of algebraic functions, in particular with integrals of rational functions of $x$ and $y$ where $y^2 = P(x)$, a polynomial of degree $n$ without multiple roots. If $n = 1$ or 2, the integration could be carried out in finite form, the result being a rational function of the values of $x$ and $y$ in the upper limit plus a sum of multiples of logarithms of such functions. Such an expression will be called logarithmico-rational in the following. But if $n$ was 3 or 4, all they could do was to reduce the integral to logarithmico-rational functions plus one or more normal forms which defied further reduction. An interesting observation made by Euler became basic for the later development: the sum of three or more integrals of the same rational function of $x$ and $y$ with upper limits $x_1, \ldots, x_k$, is a logarithmico-rational function of the upper limits provided $x_k$ is a particular rational function of $x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}$.

Order was made in this chaos by Niels Henrik Abel (1802-1829), the son of a poor Norwegian minister, himself desperately poor, shy, depressed, and ailing in health. Mathematically self-taught, he made quite a stir at the young University of Christiania (= Oslo) by his investigations on the algebraic resolution of algebraic equations. To start with he thought that he had proved that every algebraic equation could be solved by radicals, but he discovered the error and was able to use the method for a proof that the general equation of degree greater than four cannot be solved by radicals. On the basis of this discovery he was awarded a traveling fellowship which took him to Berlin in 1825 and to Paris the next year. In Berlin Crelle befriended him and got him as a collaborator for his new Journal für Mathematik in the first volume of which there are six papers by Abel, among others the paper on the non-solvability by radicals. Paris was a disappointment to him, though of some importance to his scientific development. He presented a big manuscript to the Académie des Sciences containing what was later to be called Abel's theorem. It was not printed until 1841 and then only after diplomatic representations had been made.

Abel's theorem is the direct generalization of the theorem of Euler mentioned above. Suppose that we are dealing with integrals of a fixed rational function $R(x, y)$ of $x$ and $y$ where $y$ is the root of a polynomial equation $P(x, y) = 0$ so that $y$ is an algebraic function of $x$. Again there exist sums of integrals which add up to a logarithmico-rational function of the values of $x$ and $y$ in the upper limits, provided certain relations hold between these limits. But it is no longer necessarily true that one limit is a rational function of the others. Instead there is a characteristic number $p$ of these limits which have to be algebraic functions of the other limits. This number $p$ was determined by Abel in a few cases. It was later called the genus of the curve $P(x, y) = 0$ or of the corresponding Riemann surface. It depends only on $P$ and not on $R$.

Abel did not stop at this point, however. In Paris he had become acquainted with Cauchy's work and the advantages to be gained by studying functions in the complex plane. Such considerations were not new to him, he had probably just finished his very detailed study of the binomial series for complex values of variable and exponent. He now tackled the inversion problem for the elliptic integral of the first kind, that is,

$$u = E(x) = \Sigma\int_0^x [P(t)]^{-\frac{1}{2}} dt,$$

where $P(t)$ is a polynomial of degree four. Here Euler's formula would give $E(x_1) + E(x_2) = E(x_3)$, where $x_3$ is an explicitly known rational function of $x_1$, $x_2$, $y_1$, $y_2$. Abel may have kept in mind the corresponding situation when $P(t) = 1 - t^2$. The corresponding integral,

$A(x)$ say, defines arcsin $x$ and here Euler's formula gives

$$A(x_1) + A(x_2) = A(x_1 y_2 + x_2 y_1).$$

But in this case a great simplification is obtained by introducing the inverse function $x = \sin u$ which is single-valued and Euler's formula simply becomes the addition theorem for the sine function. There was a chance that similar considerations would work for $u = E(x)$. Perhaps $x$ is a single-valued function of $u$ having an addition theorem to be read off from Euler's formula. This turned out to be the case; Abel met with complete success and the results appeared in a long memoir in volumes 2 and 3 of Crelle followed by a number of other papers some of which appeared after his death. Abel proved that inversion of the elliptic integral of the first kind led to a single-valued meromorphic double-periodic function and for this function he obtained expansions in partial fractions and as the quotient of two infinite double products. For the latter he determined representations as simple products of trigonometric functions as well as expansions in trigonometric series.

Abel met with little recognition while he was living; the university of Berlin made a professorship for him, but the call arrived a few days after his death. The Paris Academy gave him its Grand Prix also after his death.

As it so often happens, it takes a long time for an idea to ripen, but then it comes to several people at the same time. Abel's competitor was Carl Gustav Jacob Jacobi (1804-1851) whose first publication in this field also appeared in 1827, followed by an organized theory in 1829. Jacobi arrived at the same results as Abel, but he had more time to delve into the theory and his interests were also slightly different from those of Abel. Jacobi based his theory on what he called the theta functions which were essentially the trigonometric series of Abel. But in this case as well as in the rest of the theory, it was Jacobi's definitions and notation which came to be accepted.

The elliptic case having been settled, it was natural to proceed to the hyper-elliptic one: $\int R(x, y)dx$ where $y^2 = P(x)$ and the degree of $P(x)$ is at least five. Now in the elliptic case, $n = 4$, Abel and Jacobi worked with the integral $\int dx/y$ which is bounded as soon as $n$ exceeds one. But if $n = 5$ or $6$ there is one more integral of the first kind, namely $\int x\, dx/y$, so they should perhaps be considered simultaneously. Actually each integral alone is hopeless for the various determinations of the integral for a fixed upper limit are everywhere dense in the complex plane so the existence of a single-valued inverse is out of question. Guided by Abel's theorem, Jacobi finally came to a solution in 1834. Considering the equations

$$\int_0^{x_1} [P(t)]^{-\frac{1}{2}} dt + \int_0^{x_2} [P(t)]^{-\frac{1}{2}} dt = u_1,$$

$$\int_0^{x_1} t[P(t)]^{-\frac{1}{2}}dt + \int_0^{x_2} t[P(t)]^{-\frac{1}{2}}dt = u_2 ,$$

Jacobi succeeded in showing that the symmetric functions $x_1 + x_2$ and $x_1x_2$ of the upper limits are single-valued functions of the two variables $u_1$ and $u_2$ with a system of four periods. Jacobi surmised that such functions could be expressed as quotients of theta functions in the two variables $u_1$ and $u_2$. This was proved by one of Jacobi's pupils Johann Georg Rosenhain (1816-1887) in a prize memoir of 1846 published by the French Academy in 1851.

7. *Weierstrass*. Karl Weierstrass (1815-1897) was born in Ostenfelde not far from Münster in Westphalia. He had made the acquaintance of some of Steiner's work already as a high school student, but his father wanted him to enter the civil service so he was sent off to the University of Bonn to study law 1834-38. Here he became a typical Corps student, enjoying the beer and the singing and unexcelled at the Mensur (= formal duelling). Of law he learnt none, the mathematics lectures were unattractive, but he started to read Jacobi's theory of elliptic functions of 1829 by himself and decided to go into mathematics. Having heard that Christoffer Gudermann (1798-1851) lectured on Jacobi's theory at the Akademie (later university) of Münster, he finally succeeded in overcoming his father's resistance and moved to Münster in 1839 where he qualified for a teacher's certificate in 1841. Weierstrass became a high school teacher, first a year on probation in Münster, 1842-1848 at the catholic Progymnasium in Deutsch-Crone (then in West Prussia, Polish since 1919), 1848-1855 at the Collegium Hoseanum, a catholic seminary in Braunsberg (then in East Prussia, Polish since 1945); both Weierstrass and Gudermann were catholics. In 1854 Königsberg gave him a doctor's degree honoris causa. He was called to Berlin in 1856, became a member of the Berlin Academy the following year and ordinary professor at the university in 1864. In Berlin he rose to the highest fame, for about twenty-five years he was an inspiring teacher and the final authority in all questions mathematical. He was succeeded in 1892 by his pupil Hermann Amandus Schwarz (1843-1921).

Weierstrass learned two things from Gudermann: Jacobi's theory of elliptic functions and to work with power series, and he learned both lessons very well. At his own request Gudermann had assigned Weierstrass a real research problem as topic for the essay that was required for the teacher's certificate. This problem was to represent the elliptic functions as quotients of power series in the variable $u$. This he did and he returned to the problem repeatedly in later days. Ultimately it led him to the theory of the sigma functions which served as point of departure for his theory of doubly-periodic functions. The fifteen years during which Weierstrass was a school teacher, he spent very well. He read Abel's papers: "Lesen Sie Abel!" was his standing advice to his students in later years. He set himself the task of solving the inversion problem for general Abelian integrals and to put the whole investigation on a firm basis.

Now Weierstrass was a methodical, painstaking man, and a logician. The brilliant flashes of intuition, so characteristic of Abel, Jacobi and also Riemann, but rarely visited him. He distrusted intuition and tried to put mathematical reasoning on a firm basis. He was the first to criticize the naive notion of a number and to give a formal introduction of the real number system. This was never published, however, a publication by H. Kossak in 1872, claiming to contain the theory of Weierstrass, was disowned by the latter in strong terms. Incidentally, 1872 is important in the history of the real number system for in this year appeared also the papers of Georg Cantor (1845-1918), E. Heine (1821-1881), Charles Méray (1835-1911), and Richard Dedekind (1831-1916) on the same subject. Both Cantor, of set theory fame, and Heine were pupils of Weierstrass; Méray was the apostle of arithmetization and power series in France and was independent of Weierstrass.

Once the real numbers were in order, Weierstrass could proceed to build up a theory of analytic functions based upon power series and the process of analytic continuation, for one as well as for several variables. These theories were published at least in part in the memoirs of the Berlin Academy in 1876-1881 and re-issued in book form together with other material as Abhandlungen aus der Functionenlehre in 1886. These memoirs contain a number of results, now-a-days much better known than the main body of Weierstrass's work. Thus we find here the first example of a continuous, nowhere differentiable function, the first example of a power series with the circle of convergence as natural boundary, further Weierstrass's approximation theorem for ordinary and trigonometric polynomials. There is also an example of what Weierstrass called an analytic expression, representing different analytic functions in different parts of the plane. The factorization theorem for entire functions occurs in the first memoir (of 1876) forming part of the book. Weierstrass had discovered Laurent series in 1841, the following year he studied the existence of solutions of algebraic differential equations and in this connection he formulated the principle of analytic continuation.

All this material: the number system, function theory, and differential equations, was grist of the mill that was to grind out the solution of the inversion problem for Abelian integrals. A preliminary communication appeared in the Braunsberg school program of 1849, further indications were given in volume 47 of Crelle's Journal in 1853 (the reason for the honorary degree in 1854) and in a long memoir in 1856, Crelle volume 52. In 1902, long after his death, appeared a reconstruction of the whole theory, based on lecture notes, as volume IV of his Gesammelte Werke, a tome of 624 pages.

Weierstrass started the theory of entire and meromorphic functions and the serious study of power series per se. The theory of approximation goes back to him and his work led to a new era in the calculus of variations and in the theory of minimal surfaces. It was not all analysis: Weierstrass even gave a purely geometrical proof of the fundamental theorem of projective geometry. His work was finished and polished, there is nothing to correct and, within the limits set by the author, it is complete. There are no loose ends and few ideas are floating around.

8. *Riemann*. Bernhard Riemann (1821-1866) is a contrast to Weierstrass in every respect. Personally he was shy and clumsy and his health was poor, he suffered from vertigo and died young from tuberculosis. Weierstrass on the other hand was a jolly soul, quite robust, and lived to a high age. Their approach to mathematics is fundamentally different: Weierstrass had the local point of view, Riemann the global one. Weierstrass had one tool which he had perfected and used with great skill; he built slowly and with extreme care. Riemann took what tools he needed without worrying too much about them as long as the tool appeared appropriate for the purpose. His function theory is largely based upon physical and geometrical considerations. Thus the real and the imaginary parts of an analytic function are functions of the logarithmic potential and his long association with Wilhelm Weber had given him almost a physicist's point of view, the physical situation showed the existence of the functions he needed and that was enough. His geometrical intuition was highly developed and he may be regarded as one of the fathers of topology, analysis situs as he called it.

There is also quite a difference between the character of their work. Weierstrass finished what he started. Riemann on the other hand, started much more than he ever could finish; he has provided steady work for mathematicians over a century. We are still digesting and developing his ideas and trying to prove his conjectures. Just try to make a census of the concepts and problems to which the name of Riemann is attached.

You start with the Riemann integral and sums, concepts which are not quite outmoded yet. There is a Riemannian theory of trigonometric series which has led to "Riemannian" theories of other orthogonal series. We have Riemannian or geometric function theory, the Cauchy-Riemann equations, Riemann surfaces, and Riemann matrices in his theory of Abelian integrals. There is a Riemann conformal mapping problem, solved by W. F. Osgood - E. H. Taylor, P. Koebe, and C. Carathéodory around 1912-1913 for the case of simply-connected domains. There is also a Riemann problem of constructing a function holomorphic in a domain bounded by a smooth curve when the real or the imaginary part of the boundary values is given or are related by a linear equation.

The first general solution was found by David Hilbert (1862-1942) in 1904 as one of the early applications of the theory of integral equations. A more general Riemann problem was solved by G. D. Birkhoff (1884-1945) in 1913. Both Hilbert and Birkhoff applied their results to another Riemann problem: that of finding a linear differential equation with preassigned regular singular points and given monodromy group. Birkhoff also extended this to irregular singular points and to difference and $q$-difference equations. The original Riemann problem was formulated for a second order linear differential equation with three regular singular points and led to the hypergeometric equation (Riemann's fourth paper of 1857). The Riemann zeta function (considered by Euler for real variables) appeared in his seventh paper in 1859; there were originally six unproved statements or conjectures in this paper, all but the main one were proved by 1905 through the work of J. Hadamard and H. von Mangoldt, but the famous Riemann hypothesis that all complex zeros have real part ½ is still unproved. G. H. Hardy (1877-1947) in 1915 proved the existence of infinitely many zeros on this line. The prime problem in other rings leads to other zeta functions and other "Riemann hypothesies" mostly unproved. Riemann's eighth published paper on the propagation of waves in air contains the solution of a boundary problem for a wave equation involving a "function of Riemann". Finally, his habilitation lecture of 1854 on the basic hypotheses of geometry, published in 1867, gave rise to the Riemann variant of non-euclidean geometry where there are no parallels, to various analyses of the space problem, to Riemannian geometry, metric, and curvature tensor.

Winston Churchill's famous epigram "Never . . . was so much owed by so many to so few!" also accurately describes the relation between present day mathematicians and the papers of Riemann.

9. *Hermite*. It is time to turn and take a look at the development in France. Here the leading figure during the third quarter of the nineteenth cdntury is Charles Hermite (1822-1901). His most spectacular achievements were the solution of the general quintic by elliptic functions in 1858 and the proof that $e$ is transcendental in 1873. He was one of the pioneers of invariant theory and quadratic forms in France and contributed to the theory of elliptic and modular functions. Hermite's name is connected with several important concepts: he would have recognized what we call a polynomial of Hermite and an Hermitian form, but an Hermitian operator would probably have puzzled him.

Hermite's influence in the mathematical world was great. This was only partly due to family relations (his wife was a sister of J. Bertrand (1822-1900) and Émile Picard was his son-in-law); above all his fair-mindedness, urbanity and benevolent interest in younger mathematicians accounted for his power. He used his international relations and his influence in France for the good of mathematics and mathematicians. A couple of examples will illustrate this. Hermite and Weierstrass encouraged the Swedish mathematician Gösta Mittag-Leffler (1845-1927) to start the first international mathematical journal Acta Mathematica in 1882 to provide a forum where French and German mathematicians could meet and thus improve their relations which were strained after the Franco-Prussian war of 1870-71. Incidentally, Mittag-Leffler had come to Hermite to work with him in 1873, but the latter sent him off to Weierstrass with the statement that the latter was the master of all mathematicians. Another famous case is the correspondence which started in 1882 between Hermite and a young Dutch astronomer Jan Thomas Stieltjes (1856-1894) which lasted until the latter's death. Urged by Hermite to come to Paris, Stieltjes got his doctor's degree there in 1885 and a professorship was found for him in Toulouse. Stieltjes' most famous investigation, a memoir on continued fractions and the associated moment problem appeared partly after his premature death. Except for the Stieltjes integral, one of the tools he had to create for his problem, the contents of this memoir remained a sealed book for well nigh twenty-five years.

In passing let us mention some more of Hermite's contemporaries. The name of the differential geometer Ossian Bonnet (1819-1892) survives in the formulas of Gauss-Bonnet. Edmond Laguerre (1834-1886) did beautiful work in theory of entire functions, theory of equations, projective geometry where he introduced the angle as the logarithm of a cross-ratio (Laguerre polynomials were known to Abel). Émile-Léonard Mathieu (1835-1890) will be remembered for his functions, solutions of a certain differential equations with periodic coefficients, and for his group theoretic investigations. Camille Jordan (1838-1922) is famous for the Jordan curve theorem, even if he did not prove it, for the theory of functions of bounded variation, and the Jordan-Hölder theorem in group theory. Jordan's Cours d'Analyse and his Traité des substitutions were both classics and strongly influenced mathematicians of his day. The eminent differential geometer Gaston Darboux (1842-1917) also made important contributions to analysis (functions of large numbers).

10. *Poincaré*. The crop of great analysts with which France recovered the leadership in analysis started to ripen in the late seventies with Henri Poincaré (1854-1912), Paul Appell (1855-1930), Émile Picard (1856-1941), Édouard Goursat (1858-1936), later followed by Paul Painlevé (1863-1933) Jacques Hadamard (1865- ), Émile Borel (1871- ), Henri Lebesgue (1875-1941), Paul Montel (1876- ), and Maurice Fréchet (1878- ).

Poincaré was encyclopedic in his interests, a genius full of ideas and led by strong intuition, he created first class mathematics with the greatest of ease, and produced around 500 papers from 1878 until his premature death. He worked in the different branches of the theory of ordinary differential equations, real and complex, infinite determinants, continuous and discontinuous groups and non-euclidean geometry.

With Felix Klein (1849-1925) as chief competitor, he created the theory of automorphic functions during the early eighties. He was also one of the creators of modern topology (analysis situs in his terminology) to which he devoted a series of six memoirs in 1895-1904. He had been led to such questions through his work of 1880 on the shape of the integral curves of first order differential equations. He lectured on all fields of mathematical physics as well as celestial mechanics to which he contributed copiously. He wrote several books on philosophy, especially on the theory of knowledge. There is much in Poincaré's work which is sketchy and fragmentary: once he had seen through the difficulties, he had a royal disdain for pesky details.

If Sylvester was a second Adam, Poincaré tried hard to be a third, but he did not belong to the ant-school. His speciality was to name problems, functions, and concepts after famous mathematicians who were remotely connected with the idea. Thus he introduced a Neumann problem in potential theory which had never been considered by Carl Neumann (1832-1925). In one of his papers on analysis situs he introduced the Betti numbers in honor of Enrico Betti, with what other justification is unknown to me. The first class of automorphic functions which he considered was connected with the inversion problem for the quotient of two linearly independent solutions of linear second order differential equations of the Fuchsian class so he called them "fonctions fuchsiennes". When Felix Klein protested that Lasarus Fuchs (1833-1902) had never considered such functions while he, Klein, had, Poincaré promptly called the next class that he encountered "fonctions kleinéennes", because, as some wit observed, Klein had never considered them.

It is necessary to mention briefly the definition of automorphic functions. Given a group $S$ of linear fractional substitutions on the variable $z$ such that the set of transforms of $z$ under $S$ is nowhere dense in the plane. It is required to find an analytic function $f(z)$, invariant under $S$, that is, such that $f(Sz) = f(z)$ for every substitution of the group. If $S$ is finite there are rational automorphic functions, if $S$ is a group of translations periodic or doubly-periodic functions will do, but for the general situation the construction problem is rather difficult. As observed above, these functions have interesting connections with the theory of linear second order differential equations. They also arise in the problem of mapping the interior of a polygon bounded by circular arcs on the interior of a half-plane (or circle). They solve a number of uniformization problems. In particular, if $C$: $f(z, w) = 0$ is an algebraic curve, then it is possible to find automorphic functions $G(t)$, $H(t)$ such that $w = G(t)$, $z = H(t)$ is a parametric representation of $C$ by means of single-valued functions meromorphic in their domain of existence.

11. *Picard.* Let us now pass over to Émile Picard who made his first striking discovery in 1879, published in detail in the Annales de

l'École Normale Supérieure, Series 2, volume 9 (1880). Here Picard showed that an entire function can omit at most one finite value without reducing to a constant, and if there exist two values each of which is taken on only a finite number of times, then the function is a polynomial. If the function is meromorphic instead, infinity being an admissible value, at most two values can be omitted without the function reducing to a constant. In the same paper he extended the conclusions to the roots of the equations $F(z) = a$, $F(z) = b$ in the neighborhood of an essential singular point. The proof is based on properties of the modular function and its inverse. A few years later (1883, 1887), Picard proved that if $f(z,w) = 0$ is an algebraic curve of genus greater than one, then it is not possible to satisfy the equation by two single-valued functions $w = G(t)$, $z = H(t)$ having isolated essential singular points, in particular not by functions meromorphic in the finite plane. Picard also studied discontinuous groups in space, algebraic integrals of several variables, partial differential equations, integral equations, the extension of the Galois theory to differential equations, and various questions of mathematical physics.

Picard's theorem exercized a tremendous influence on the development of analysis which is still lasting. For a good many years attempts were made to obtain a simpler proof without the use of the modular function. This was achieved by Émile Borel in 1896 and led to very surprising consequences, in particular, to a theorem of Edmund Landau (1877-1938) of 1904 according to which there is a circle whose radius depends only on the first two coefficients of a given power series such that the function defined by the series either takes on the value zero or the value one or has a singular point inside or on the circle. This was considered a great triumph of the elementary method, but the following year Constantin Carathéodory (1875-1950) showed that the correct value of the radius was expressible in terms of the modular function. The whole question of exceptional values was put on a much more general basis through the investigations of the brothers Frithiof and Rolf Nevanlinna (born 1894 and 1895 respectively). These investigations, having started in 1922, lie outside the scope of this paper, however.

The nineties and the following decade saw a number of investigations devoted to the theory of entire functions, relations between the rate of growth of the maximum modulus, the frequency of the zeros, and the decrease of the coefficients. A multitude of names could be mentioned, we restrict ourselves to those of Borel, Hadamard, Ernst Lindelöf (1870-1946), and Edvard Phragmén (1863-1937). The latter are chiefly famous for an extension of the maximum principle proved in 1908 and based on previous work by Phragmén alone in 1904. It is one of the most powerful tools available in function theory. G. Mittag-Leffler had started a very successful school of mathematicians at the new University of Stockholm in 1882; Phragmén belonged to his early pupils and also held a professorship there before going into life insurance where he held a number of influential positions and was instrumental in establishing the close relations between research mathematics and the life insurance companies which is characteristic for Scandinavian conditions. Lindelöf as professor at the University of Helsingfors founded a flourishing mathematical school in Finland. The Nevanlinnas mentioned are among his most distinguished pupils.

12.*Hilbert*. The turn of the century saw two basic discoveries in analysis which were to dominate the future development, one was the Lebesgue integral presented in his dissertation of 1902 and the first of a number of integrals, each more abstract than the previous one. Lebesgue's own work on Fourier series having demonstrated the usefulness of the new notion, it gradually penetrated all branches of real analysis. A contributing factor of the first order of magnitude was the work by F. Riesz [(1880- ) Riesz Frigyes in Hungarian, Frederick, Frédéric or Friedrich Riesz according to the language of the paper] on functions integrable together with their squares or $p$th powers (1907, 1910). Through this work on Lebesgue spaces, on functionals of continuous functions, and his book on linear equations in infinitely many unknowns (1913), F. Riesz influenced the whole development of what became known as functional analysis.

The second discovery was the theory of integral equations, one of the forerunners of functional analysis and the theory of operators. Integral equations with variable upper limits occur in the works of Abel and of Liouville, but the first general theory is due to Vito Volterra (1860-1940) starting in 1896. Volterra also started the theory of functionals. Integral equations with a constant interval of integration were the creation of Mittag-Leffler's most famous pupil Ivar Fredholm (1866-1927) by a preliminary communication in 1900 and a finished theory in 1903. Fredholm was led by the analogy with systems of linear equations, but instead of carrying out the limiting process he boldly wrote down the resulting infinite determinants and verified that they satisfied. It is no accident that Fredholm used infinite determinants for another of Mittag-Leffler's pupils Helge von Koch (1870-1924) had developed this theory to a perfection.

The further development was taken over by David Hilbert (1862-1942). Hilbert, one of the giants of our science, was born in Königsberg in East Prussia where he got his doctor's degree in 1885. His early work was devoted to the theory of forms and invariants. In 1892 Hilbert turned to algebraic number theory, culminating in his enormous report on algebraic fields to the Deutsche Mathematiker Verein in 1897. Hilbert had a knack of changing his interests in mathematics suddenly and completely: in 1899 appeared his "Grundlagen der Geometrie" which led to a revival of the postulational method, first in geometry, later in all other fields. The same year he turned away from geometry and was preparing an "Ehrenrettung" of Dirichlet's principle in the calculus of variations. These papers belong to the period 1899-1906. Hilbert had moved from Königsberg to Göttingen in 1895 where Felix Klein represented the Riemannian tradition in function theory. After Weierstrass's very justified criticism of Dirichlet's principle, the Riemann school lacked a basis for their existence theorems and it was essential that somebody put the theory back on a firm basis. This Hilbert succeeded in doing, much to the relief of his colleagues.

An accidental seminary lecture (by E. Holmgren of Uppsala) on Fredholm's work directed Hilbert's attention to the field of integral equations in 1901 and he now turned most of his energy into new channels. He created the theory of integral equations with a symmetric kernel, the prototype of all theories of symmetric operators, and showed how it tied in on one hand with a theory of quadratic forms in infinitely many unknowns and on the other with the theory of complete orthogonal systems and the question of expansions in terms of such systems. This period lasted until about 1912, during the latter part of which Hilbert used the new tools for an attack on the problems of mathematical physics.

To this period belongs also Hilbert's solution of Waring's problem on the representation of integers as sums of $n$th powers in 1909. Hilbert gave a mere finiteness theorem and the actual determination of limits for the necessary number of powers required was left for the powerful analytical methods invented in 1916 by G. H. Hardy and J. E. Littlewood (1885- ).

Another of the great events at the beginning of the century was the formulation of the axiom of choice by Ernst Zermelo (1871- ) in 1904 and his proof of the well-ordering theorem based on this axiom. It would take us too far to follow the vicissitudes of this axiom and the closely related conflict between formalists and intuitionists in mathematical logic which occupied much of Hilbert's attention during the last twenty years of his life.

Yale University

# TEACHING OF MATHEMATICS

*Edited by*

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## OFF THE BEATEN PATH WITH SOME DIFFERENTIATION FORMULAS FOR THE TRIGONOMETRIC, EXPONENTIAL AND LOGARITHMIC FUNCTIONS

J. C. Eaves

1. Introduction. It is generally conceded to be bad pedagogy as well as bad psychology to ignore completely the text adopted for use in an elementary course. However, if a deviation from the "cut and dried" presentations usually employed simplifies the work and makes for better and quicker understanding by the student then such a deviation is certainly justifiable. I believe that the following rearrangement of the order of development of the formulas for differentiation of the trigonometric, exponential and logarithmic functions can be effected with a minimum of confusion to the student in his "follow the book" routine. This approach has been found, by the author, to offer several advantages over the orthodox method. I present the outline here in the hope that others may find it helpful in teaching these topics in an introductory course.

2. The trigonometric functions. Formulas for differentiating the trigonometric function are usually derived for these functions in the following order: $\frac{d(\sin u)}{dx}, \frac{d(\cos u)}{dx}, \frac{d(\tan u)}{dx}, \ldots$ . Everyone is familiar with the confusion arising when, in the development of $\frac{d(\sin u)}{dx}$, the double angle formula is introduced. Then, after a couple of substitutions and other "mathematical gymnastics", one most often finds a statement similar to, "with a little rearrangement this becomes", followed all too often with an expression very much like

$$\frac{\Delta y}{\Delta x} = \left\{\cos u \cdot \frac{\sin \Delta u}{\Delta u} - \sin u \cdot \frac{\sin \frac{\Delta u}{2}}{\frac{\Delta u}{2}} \cdot \sin \frac{\Delta u}{2}\right\} \frac{\Delta u}{\Delta x},$$

much to the bewilderment of the student.

The development of $\dfrac{d(\cos u)}{dx}$ affords the student as much confusion.

Let us obtain the differentiation formulas in the order: $\dfrac{d(\tan x)}{dx}$, $\dfrac{d(\sec x)}{dx}$, $\dfrac{d(\sin x)}{dx}$, . . . Having argued (and I use the word "argued" because a rigorous proof is usually omitted in an elementary text) that $\lim\limits_{\alpha\to 0}\dfrac{\sin \alpha}{\alpha} = 1$ for $\alpha$ measured in radians, it then follows that $\lim\limits_{\alpha\to 0}\dfrac{\tan \alpha}{\alpha} = 1$ for $\alpha$ measured in radians.
Then, *briefly*, for

(i) $$y = \tan x$$

We have $$\Delta y = \tan (x + \Delta x) - \tan x$$

$$= \frac{\tan x + \tan \Delta x}{1 - \tan x \tan \Delta x} - \tan x$$

$$= \frac{\tan x + \tan \Delta x - \tan x + \tan^2 x \tan \Delta x}{(1 - \tan x \tan \Delta x)}$$

Thus, $$\frac{\Delta y}{\Delta x} = \frac{(\tan \Delta x)(1 + \tan^2 x)}{(\Delta x)(1 - \tan x \tan \Delta x)}$$

and $$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = (1 + \tan^2 x) = \sec^2 x.$$

Next, developing the formulas for $\dfrac{d(\sec x)}{dx}$, we have briefly,

(ii) $$y = \sec x$$

$$y^2 = \sec^2 x = 1 + \tan^2 x.$$

Then, $$2y\frac{dy}{dx} = 2 \tan x \sec^2 x$$

and $$\frac{dy}{dx} = \sec x \tan x.$$

By writing $y = \sin x = \dfrac{\tan x}{\sec x}$, $y = \cos x = \dfrac{1}{\sec x}$, $y = \csc x = \dfrac{1}{\sin x}$, and $y = \cot x = \dfrac{1}{\tan x}$ and differentiating these quotients, the student can easily and quickly obtain the desired formulas.

In order to avoid the psychological effect of "skipping", "omitting", or "doing something not in the text, and which the student can't understand", the best approach seems to be that in which the functions $y = \sin x$, $y = \cos x$, $y = \tan x$ are all written on the board when the trigonometric functions are to be considered. Occasionally someone who has looked over the development of $\frac{d(\sin x)}{dx}$ and $\frac{d(\cos x)}{dx}$ suggests, "Let's take $\tan x$," probably with the hope that he may be able to follow it. If you get no voluntary suggestions, then perhaps the remark, "Let's see which one of these we can 'push thru' in a hurry," will do the trick. As soon as any one approach seems to bog down, try another function. Over a period of several classes, more than ninety per cent of my students have selected the development of $\frac{d(\tan x)}{dx}$ by the delta method as being the simplest initial formula to obtain. Every step in its development seems to suggest itself, i.e., sum formula, addition of fractions, factoring, $\lim\limits_{\Delta x \to 0} \frac{\tan \Delta x}{\Delta x}$ enters naturally, etc.

Of course, one may use the function $y = \tan u$, where $u$ is a function of $x$; however, there seems to be no need to introduce the chain rule unless some use is to be made of it. The chain rule theorem seems to be one theorem which is completely underworked.

3. The exponential and logarithmic function. It is usually customary to present the differentiation of the logarithmic function ahead of that of the exponential function. Most texts contain either the function $y = (1 + \frac{1}{x})^x$ or the function $y = (1 + x)^{\frac{1}{x}}$, and a table of values for $x$ and $y$. Following some six or eight pairs of values for $(x,y)$ is found a phrase very similar to:

"Therefore it is reasonable to suppose that . . ."
"Then it is plausible that . . ."
"It is now clear to you that . . ."
"It is then, almost obvious that . . ."
"We consequently assume that . . ."
"It is well known that . . ."

$\lim\limits_{u \to 0} (1 + u)^{\frac{1}{u}} = 2.7\ 1828\ 1828\ 4590\ 4523\ 536\ . . .$ (well, perhaps not as far as this).

That this limit could exist was neither obvious nor even quite clear to me when I first encountered this topic in the calculus. I am still unable to list this limit as one of those among the immediately evident.

Today, many students studying the calculus have already encountered infinite series in their work in applied fields, particularly in physics. They are usually familiar with the idea of convergence and divergence thru the much used geometric series and harmonic series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots ,$$

and

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots .$$

The series

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is not new to most of these students. They have used it. They understand some of its properties. It is easy to get a discussion started on its behavior. If this series is new to some, it is obvious to them that if we are to use it and refer to it often, then it will be convenient to have a short name for it, say John, Joe, or Schluf. Of course, it will be less confusing if we decide to call it by the same name that mathematicians, physicists, and others call it, namely, $e^x$.

In some classes I have developed rather extensively the theory in connection with the convergence, differentiation and general exponential properties of $e^x$, while in other classes I have merely defined $e^x$ in the series form and then stated that the series does converge, can be differentiated term by term and in general behaves in a most obliging and desirable manner. The extent to which the theory is developed will differ from class to class and the instructor should be guided by the caliber of his students.

I do not propose to give a rigorous development of the exponential series in every class. Even when the discussion was very limited I found this approach to be more easily accepted by students who usually are not interested in rigor and, in fact, who probably could neither appreciate nor follow a rigorous treatment of this topic. I have found it to be more obvious to more students than is the conventional treatment which is, intentionally, nonrigorous.

After some necessary, altho in most cases somewhat brief, discussion the development of $\frac{d(e^x)}{dx}$ follows easily, for, by definition, we write

(iii) $$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots .$$

Then, from $y = e^x$ it follows that

$$\frac{dy}{dx} = 0 + \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots .$$

Thus,

$$\frac{dy}{dx} = e^x .$$

Occasionally, a student states that he feels that we are always behind one term in $\frac{d(e^x)}{dx}$. However, it is usually easy to convince him that in either case he will get "approximately" the same thing if he adds the first thousand or hundred thousand non-zero terms of $e^x$ or $\frac{d(e^x)}{dx}$.

It is my opinion that much is to be gained by reversing the usual order of presenting the derivative of the logarithmic function prior to the derivative of the exponential function. For one thing, it is now clear just why it is advantageous to use such an "animal" as $e$ for the base of the natural logarithms. Putting $x = 1$ we get

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

It is this very $e$ which we use as the base of the natural logarithm. This seems to be an appropriate place to introduce $e$ as a base of a system of logarithms.

By using the definition of a logarithm and the differentiation formula (iii) the derivative of the log function follows readily. Briefly we have

(iv) $$y = \log_e x.$$

That is, $$x = e^y.$$

Then, $$\frac{dx}{dy} = e^y.$$

Thus $$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

The following problems seem to form a good sequence of illustrations and generalizations to follow these two simple formulas. It seems better to reduce each of these to a problem involving $e^u$ or $\log_e u$, or $v \log_e u$, etc. The existence of these functions should be discussed.

(1) $y = a^x$, then $\log y = x \log a$, etc.

(2) $y = e^v$, $v$ a function of $x$.

(3) $y = a^v$, $v$ a function of $x$.

(4) $y = \log_e v$.

(5) $y = \log_a x$.

(6) $y = \log_a v$, $v$ a function of $x$.

(7) $y = x^x$.

(8) $y = \log \log x$.

(9) $y = (\sin x)^{\log x}$.

(10) $y = \log x^x$.

(11) $y = \log^x x$.

(12) $y = \log_x a$.

(13) $y = \log_x e$.

(14) $y = u^v$, $u$ and $v$ are functions of $x$.

(15) $y = x^u$.

(16) $y = u^u$.

(17) $y = \log_v u$.

(18) $y = \frac{u}{v}$, $u$ and $v$ are functions of $x$.

(19) $y = uv$, $u$ and $v$ are functions of $x$.

This treatment was presented to the College Mathematics Teachers of Alabama at the Auburn meeting, April 1951. The author has profited greatly from suggestions made by Dr. G. W. Hess, Howard College, Birmingham. Several mathematics teachers have told me that they had either thought of using a similar approach or that they had wondered why someone had not presented or advocated such a treatment. Perhaps the reader has either used or has thought of using it.

Various rigorous treatments of the logarithmic and exponential functions can be found in some calculus books. For example: Courant, de la Vallee Poussin, Osgood, to name a few. At this writing, a new text by Fort is in press. According to a press announcement by the publisher his opening chapter is on infinite series. I have seen no reference to the development of $\frac{d(\tan x)}{dx}$ by the delta method.

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## ON ARITHMETIC METHODS IN ELEMENTARY ALGEBRA

B. E. Mitchell

Many problems in college algebra texts have been pilfered from old arithmetics such as J. W. Nicholson's "Advanced Arithmetic", University Publishing Company, New Orleans, 1889. The writer usually presents the arithmetic methods of solution to his classes, in addition to the textbook processes, since he feels the simplest solution is the best one. If the purpose of a problem is merely to give drill in algebraic techniques, then the problem should be chosen so that these techniques are the best way to handle it. For instance, various mixture problems, when using the ordinary algebraic methods, lead to a linear equation in one unknown, two linear equations in two unknowns, two equations linear in the reciprocals of two unknowns, etc. However, nearly all these problems can be solved more easily by a simple application of proportion. Consider the following typical problem which can be readily solved mentally.

> "A 21 quart capacity car radiator is filled with an 18% alcohol solution. How many quarts must be drained and then replaced by a 90% alcohol solution for the resulting solution to contain 42% alcohol?"

One quart of the old solution differs from one quart of the new (or average) solution by -24%, while one quart of the solution to be added differs from the new solution by +48%. Hence we must have two quarts of the old solution for each quart of added solution. That is, we would drain 1/3 of the radiator of 7 quarts.

This same method may also be applied to certain systems of linear diophantine equations. It is a tentative process, but then so is long division. Consider this hoary problem.

> "A farmer must buy 100 head of animals with \$100. If calves cost \$10 each, lambs cost \$3 each, and pigs cost \$.50 each, how many of each does he buy?"

The average cost per animal is \$1. Each calf differs from the average price by +\$9, each lamb by +\$2, and each pig by -\$1/2. Hence for each calf he must have 18 pigs, and for each lamb he must have 4 pigs. A trial or two shows that he must buy 5 calves and 1 lamb and hence 94 pigs in order to have a total of 100 animals.

Another standard type of problem is to find the rational number whose decimal expansion is a given periodic decimal. This is usually solved by use of the summation formula for geometric series. The arithmetic method is illustrated by an example.

Let $d = .123454545\ldots$ .

Then $1000\,d = 123.454545\ldots$ ,

and $10\,000\,d = 12345.454545\ldots$ .

So $$9\,000\,d = 12222 \quad \text{and}$$
$$d = 12222/9000\ .$$

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## ON THE FRENCH METHOD OF LONG DIVISION

B. E. Mitchell

Long division is a process that is seldom utilized at the present time. However on occasion, as in Mathematics of Finance courses, it cannot be avoided. When it is necessary to perform a long division, some time can be saved by choosing the French method in which only the remainders are written down at each stage. The secret is not great mental agility but simply the Austrian method of subtraction which "carries on the bottom row instead of "borrowing" on the top row as we ordinarily do. Both methods are illustrated by examples.

(i) Subtract 986 from 1737. The work is here: $\begin{array}{r}1737\\ 986\\ \hline 751\end{array}$ . We say 6 and 1 are 7 and write 1; 8 and 5 are 13, write 5 and carry 1; 9 and 1 make 10, 10 and 7 are 17, write 7 and carry 1; 1 and 0 are 1. This method eliminates subtraction tables.

(ii) Divide 25456 by 236. The work is shown below:

```
           107
  236 / 25456
         1856
          204
```

We say 1 × 6 is 6 and 8 makes 14 (adding the smallest possible multiple of 10 to 4), write 8 and carry 1; 1 × 3 is 3 and 1 makes 4, 4 and 1 make 5, write 1; 1 × 2 is 2 and 0 makes 2. Bring down the 5. Since 236 will not divide 185 we bring down the 6 also. Then we say 7 × 6 is 42 and 4 makes 46, write 4 and carry 4; 7 × 3 is 21 and 4 makes 25, 25 and 0 make 25, write 0 and carry 2; 7 × 2 is 14 and 2 makes 16, 16 and 2 make 18, write 2. The remainder then is 204.

These methods are taught not only in France but also in the Commercial schools of Germany. Although they are properly concepts of elementary schools, they do not seem to be well known in this country. The writer believes it would be an improvement if these methods were introduced in our elementary schools.

Alabama Polytechnic Institute

# MISCELLANEOUS NOTES

*Edited by*

Charles K. Robbins

Articles intended for this department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

Dear Dr. James,

I need some assistance to give evidence that the peace of this world is damned because of the dogma set forth in "Fundamental Concepts of Algebra and Geometry" by J. W. Young.

Here is the final criterion I need to solve the 3-body problem completely. If this criterion prŏves true then the problem of Peace is easily solved in the same way that I have solved this mathematical problem.

Using the notation of Moulton's "Celestial Mechanics", 1902 edition, pages 262-278 and letting the elements for the case of plane motion be $\beta_i$ = $a$, $e$, $\epsilon$, and $T$; the solution depends upon having $f = \frac{1}{e} - \frac{pk^2(m_1 + m_2)}{eh^2}$ make the last equation given herewith an identity. If it should not be an identity then there is no solution of this form to either problem.

The equations are:

$$\theta = v + \omega; \ \frac{\partial\theta}{\partial t} = \theta = \frac{h}{r^2}; \ \frac{\partial v}{\partial t} = \frac{gh}{r^2}, \ \frac{\partial\omega}{\partial t} = \frac{h}{r^2}(1 - g);$$

$$\sigma = \rho T = \epsilon - \omega; \ \frac{\partial\rho}{\partial t} = -\frac{\partial\omega}{T\partial t} = \frac{h(g - 1)}{Tr^2}$$

$$\frac{\partial g}{\partial t} = \frac{h \cot v}{r^2}\{1 + f \sec v - g^2\}$$

These partials may be looked upon as assumptions, so that one needs only see to it that they are used consistently. Then

$$r = a(1 - e \cos E) = \frac{p}{1 + e \cos v}$$

$$\frac{\partial r}{\partial t} = \dot{r} = ae \sin E \frac{\partial E}{\partial t} = \frac{r^2 e \sin v}{p}\frac{\partial v}{\partial t} = \frac{geh}{p} \sin v;$$

$$\frac{\partial E}{\partial t} = -\frac{gh}{ar\sqrt{1 - e^2}} ;$$

$$\rho(t - T) = E - e \sin E$$

$$\frac{\partial \rho}{\partial t}(t - T) + \rho = \frac{r}{a}\frac{\partial E}{\partial t} = -\frac{gh}{a^2\sqrt{1 - e^2}}$$

So that
$$\rho = \frac{gh}{a^2\sqrt{1 - e^2}} + \frac{h(t - T)}{T' r^2}(1 - g)$$

Now find $\frac{\partial \rho}{\partial t}$ again – i.e.

$$\frac{\partial \rho}{\partial t} = h\left[\frac{1}{a^2\sqrt{1 - e^2}} - \frac{(t - T)}{Tr^2}\right]\frac{\partial g}{\partial t} + \frac{h(1 - g)}{Tr^2} - \frac{2\dot{r}h(t - T)(1 - g)}{Tr^3} = \frac{h(g - 1)}{Tr^2}$$

eliminate $\frac{\partial g}{\partial t}$ and $\dot{r}$, and solve for $g$; then find $\frac{\partial}{\partial t}(\ )$ of resulting equation from which $\frac{\partial g}{\partial t}$, $\dot{r}$, and $g$ can be eliminated. This is the equation which must now be an identity when $f = \frac{1}{e} - \frac{k^2(m_i + m_j)p}{eh^2}$ is substituted.

In finding the partials $\frac{\partial(\ )}{\partial t}$ one should remember that $\frac{\partial \beta_i}{\partial t} = 0$ and that $h$, $p$, and $f$ are functions of $\beta_i$ (the elements) only so that $\frac{\partial h}{\partial t} = \frac{\partial p}{\partial t} = \frac{\partial f}{\partial t} = 0$.

Could you see the way clear to assist me in verifying that this equation reduces to an identity?

You might assist me by printing this letter.

Sincerely Yours,

Glen H. Draper
U. S. Naval Clbs.
Washington 25, D.C.

Will some one please assist Mr. Draper in this problem? Editor.

# PROBLEMS AND QUESTIONS

*Edited by*

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction. Readers are invited to offer heuristic discussions in addition to formal solutions.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

## PROPOSALS

**154.** *Proposed by F. L. Miksa, Aurora, Ill.*

Find primitive Pythagorean triangles each of whose areas is equal to some permutation of the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. [Dedicated to Victor Thébault.]

**155.** *Proposed by Leon Bankoff, Los Angeles, Calif.*

An equilateral triangle is circumscribed by a chain of six equal and successively tangent circles in such a manner that three of the circles touch the vertices and the other three the midpoints of the sides of the triangle. Show that the inradius of the triangle is equal to the side of a regular decagon inscribed in one of the tangent circles.

**156.** *Proposed by E. P. Starke, Rutgers University.*

A quadrilateral has sides of length 2, 3, 4, 5 in that order. Determine the angle between the sides of length 4 and 5 such that the area shall be the largest possible integer.

**157.** *Proposed by W. H. Glenn, Jr., Pasadena City Schools, Calif.*

In Woods and Bailey, *Analytic Geometry and Calculus*, Ginn (1944), page 365, problem 69 states: "Through a given point (1, 1, 2) a plane is passed which, with the coordinate planes, forms a tetrahedron of minimum value. Find the equation of the plane." A student set up the volume as $V = abc/6$, where $a$, $b$, $c$ are the $x$, $y$, $z$ intercepts of the plane through (1, 1, 2). He then took $\partial V/\partial a$, $\partial V/\partial b$, $\partial V/\partial c$ and evaluated them at $a = 1$, $b = 1$, $c = 2$. He then said that the plane he was seeking was

$$(\partial V/\partial a)_P(x - 1) + (\partial V/\partial b)_P(y - 1) + (\partial V/\partial c)_P(z - 2) = 0,$$

where the subscript $P$ indicates that the partial derivative has been evaluated at $P(1, 1, 2)$.

Show that this result is correct and that the procedure can be generalized to $P(x_1, y_1, z_1)$.

**158.** *Proposed by Leo Moser, University of Alberta, Canada.*

Find all values of $r$ such that no $n!$ (written in decimal notation) can end in exactly $r$ zeros.

**159.** *Proposed by A/2C D. L. Silverman, Patrick AFB, Florida.*

A man breaks a stick in two places. What is the probability that he will be able to form a triangle with the three segments?

**160.** *Proposed by Victor Thébault, Tennie, Sarthe, France.*

In which of the remaining years of the twentieth century will Friday-the-thirteenth occur most (or least) frequently?

## SOLUTIONS

### A Non-unique Cryptarithm

**128.** [March 1952] *Proposed by Victor Thébault, Tennie, Sarthe, France.*

In the division transformation $ABCD \times E + F = GHIJ$, no two letters represent the same digit. Identify the letters.

*Solution by F. L. Miksa, Aurora, Ill.* Clearly there is no solution for $E = 1$, for then $A = G$. Also, for $E = 9$, $ABCD = (GHIJ - F)/E \leqq 8765/9 < 1000$, so again there is no solution. For $E = 8$, $9008/8 = 1126 < ABCD < 1221 = 9768/8$, hence $1203 \leqq ABCD \leqq 1209$. In like manner, limiting values of $ABCD$ can be established for the other possible values of $E$. Searching within the ranges thus established we find the following twenty-one solutions:

| | | |
|---|---|---|
| $1749 \times 2 + 8 = 3506$ | $4307 \times 2 + 5 = 8619$ | $1269 \times 4 + 7 = 5083$ |
| $3407 \times 2 + 5 = 6819$ | $4671 \times 2 + 8 = 9350$ | $1273 \times 4 + 6 = 5098$ |
| $3471 \times 2 + 8 = 6950$ | $4761 \times 2 + 8 = 9530$ | $1826 \times 4 + 5 = 7309$ |
| $3519 \times 2 + 8 = 7046$ | $1806 \times 3 + 9 = 5427$ | $1304 \times 6 + 5 = 7829$ |
| $3549 \times 2 + 8 = 7106$ | $2059 \times 3 + 7 = 6184$ | $1342 \times 6 + 7 = 8059$ |
| $4067 \times 2 + 5 = 8139$ | $2157 \times 3 + 9 = 6480$ | $1403 \times 7 + 5 = 9826$ |
| $4079 \times 2 + 5 = 8163$ | $2605 \times 3 + 4 = 7819$ | $1204 \times 8 + 5 = 9637$ |

Also solved (partially) by *R. E. Crane, Morristown, N.J.*

### Intersecting Congruent Ellipses

**129.** [March 1952] *Proposed by E. P. Starke, Rutgers University.*

Two equal ellipses are at first in coincidence. Then one of them is rotated about their common center through such an acute angle $\phi$ that each has just half its area in common with the other. Show that $\sin\phi = 2ab/(a^2 - b^2)$ where $a$, $b$ are the lengths of the semi-axes.

*Solution by J. A. Tierney, U. S. Naval Academy.* In polar coordinates the equations of the ellipses are

$$r^2 = a^2b^2/(b^2\cos^2\theta + a^2\sin^2\theta)$$

and

$$r^2 = a^2b^2/[b^2\cos^2(\theta - \phi) + a^2\sin^2(\theta - \phi)].$$

Solving simultaneously we have $\sin\theta = \pm\sin(\theta - \phi)$, so that two consecutive solutions are $\theta = \phi/2$ and $\theta = \phi/2 + \pi/2$, as was to be expected from the congruence of the ellipses. It follows that one fourth of the common area is

$$\frac{1}{2}a^2b^2\int_{\phi/2}^{\phi/2+\pi/2} d\theta/(b^2\cos^2\theta + a^2\sin^2\theta) = \pi ab/8.$$

Integrating,

$$ab[(1/ab)\arctan(a\tan\theta/b)]_{\phi/2}^{\phi/2+\pi/2} = \pi/4$$

or

$$\arctan(-ab^{-1}\cot\phi/2) - \arctan(ab^{-1}\tan\phi/2) = \pi/4.$$

Taking the tangent of both sides of the equation, we have

$$(-ab^{-1}\cot\phi/2 - ab^{-1}\tan\phi/2)/(1 - a^2/b^2) = 1.$$

Now $\cot\phi/2 + \tan\phi/2 = (1 + \cos\phi)/\sin\phi + (1 - \cos\phi)/\sin\phi = 2/\sin\phi$, so

$$\sin\phi = 2ab/(a^2 - b^2).$$

In order for a solution to exist we must have $\sin\phi \leqq 1$, which holds whenever $a \geqq (1 + \sqrt{2})b$.

Also solved by *Howard Eves, Champlain College; A. Sisk, Maryville, Tennessee; and the proposer.*

### A Prime Number Property

**132.** [March 1952] *Proposed by Samuel Skolnik, Los Angeles City College.*

Prove that if $P_n$ is the $n$th prime (in the order of magnitude), then for any $\epsilon > 0$, there exists a positive integer $N$ such that $n^{1+\epsilon} > P_n$ for all $n > N$.

*Solution by the proposer.* By the prime number theorem, $n/[P_n/\log P_n] = 1 + \delta_n$ or $n = P_n(1 + \delta_n)/\log P_n$ where $|\delta_n| \to 0$ as $n \to \infty$. Hence there exists a positive integer $N_1$ such that

$$n > P_n/2 \log P_n \text{ for all } n > N_1. \qquad (1)$$

Now

$$\lim_{n\to\infty} \frac{P_n/2 \log P_n}{P_n^{1/(1+\epsilon)}} = \lim_{n\to\infty} \frac{P^{\epsilon/(1+\epsilon)}}{2 \log P_n} = \lim_{n\to\infty} \frac{[\epsilon/(1+\epsilon)]P^{-1/(1+\epsilon)}}{2/P_n}$$

$$= \lim_{n\to\infty} \frac{\epsilon P_n^{\epsilon/(1+\epsilon)}}{2(1+\epsilon)} = \infty.$$

Consequently

$$P_n/2 \log P_n > P_n^{1/(1+\epsilon)} \text{ for all } n > N_2. \qquad (2)$$

Thus from (1) and (2) we have $n > P_n^{1/(1+\epsilon)}$ for all $n > N$, where $N$ is the larger of $N_1$ and $N_2$. Finally,

$$n^{1+\epsilon} > P_n \text{ for all } n > N.$$

## A Diophantine System

**133.** [May 1952] *Proposed by W. R. Talbot, Jefferson City, Missouri.*

If $a$, $b$, $c$ and $d$ are used to replace distinct non-zero digits, find their values in the equations

$$(ca)^2 + (ab)^2 = (cb)^2 + (c^2)^2 = (cc)^2 + (d)^2 = (bd)^2 + (bc)^2.$$

I. *Solution by J. H. Monahan, Boston College.* Call the members (1), (2), (3), (4) in the order given. From (1) and (3), $d^2 = (ca)^2 - (cc)^2 + (ab)^2$. Now if $c = 1$, $d^2 \geqq (12)^2 - (11)^2 + (23)^2 = 552$. If $c = 2$, $d^2 \geqq (21)^2 - (22)^2 + (13)^2 = 126$. But $d$ is a digit so $81 \geqq d^2$. Hence $c \neq 1$ or 2.

From (2) and (3), $(cb)^2 + (c^2)^2 - (cc)^2 = d^2$. Now if $c = 4$, $b^2 + 80(b-1) = d^2$. Further, if $b = 1$, $d = 1$; and if $b \geqq 2$, the left member exceeds 81. Hence, $c \neq 4$.

Now make the left member of the equation from (2) and (3) as small as possible, with $c$ remaining a variable, by letting $b = 1$. The equation then assumes the form

$$c(c-1)(c-4)(c+5)+1=d^2.$$

If $c = 5$, the value of the left member is 201, and as $c$ increases so will the value of the left member. But its value was made as small as possible, so it will always exceed 81 for $c \geqq 5$. Thus $c$ is not greater than 4.

The only remaining possible value for $c$ is 3. For this value the equation from (2) and (3) assumes the form $b^2 + 60b - 108 = d^2$. For $b = 1$, the left member is negative; for $b > 3$, the left member exceeds 81; for $b = 3$, $d = 3$; and for $b = 2$, $d = 4$. Placing the last pair of values in the equation from (1) and (3), we have $(101a + 201)(a - 1) = 0$. Hence the unique solution is $a = 1$, $b = 2$, $c = 3$, $d = 4$.

It will be observed that member (4) has not been employed in this solution and hence is superfluous. That it is satisfied by the values found is seen upon checking the original expression. That is,

$$(31)^2 + (12)^2 = (32)^2 + (9)^2 = (33)^2 + (4)^2 = (24)^2 + (23)^2 = 1105.$$

**II.** *Solution by Erich Michalup, Caracas, Venezuela.* From (3) and (4) we have $121c^2 + d^2 = 200b^2 + 20b(c+d) + d^2 + c^2$ or $10b^2 + b(c+d) - 6c^2 = 0$. Since $b$ is positive, it follows that

$$b = [-(c+d) + \sqrt{(c+d)^2 + 240c^2}]/20 \text{ or } [-(c+d) + r]/20,$$

where $r$ is an integer. Now $b$, $c$, $d$ are distinct digits so the only solutions of the last expression are $(b, c, d) = (2, 3, 4)$ and $(4, 6, 8)$, to which correspond the values 1105 and 4420, respectively, for each of the members. Now the corresponding values of (2) are 1105 and 5392. Thus (4, 6, 8) is eliminated from consideration. Equating (1) to 1105 we have $(30 + a)^2 + (10a + 2)^2 = 1105$ or $(101a + 201)(a - 1) = 0$. Hence the unique solution is $(a, b, c, d) = (1, 2, 3, 4)$.

**III.** *Solution by the Proposer.* From (3) and (4), since $bd > d$, then $cc > bc$, so $c > b$. Then from (1) and (4), $ca > bd$, so $bc > ab$ and $b > a$. Then from (1) and (2), $cb > ca$, so $ab > c^2$. It follows that $b < 9$, $a < 8$ and $c > 2$.

From (1) and (2), $a^2 + b^2$ has the same last digit as $b^2 + c^4$, so $a^2$ and $c^4$ have the same terminal digit. Now all fourth powers of non-zero integers except 5 end in 1 or 6, and we cannot have $c = 5$ lest $a = 5$. Hence the terminal digit is 1 or 6.

If $c^4$ ends in 6, $a$ might be 4 or 6 with $c = 6$ or 8, or 8, respectively. With $a = 4$ and $c^2 < ab$ we have $c = 6$. Then since $c > b > a$, $b = 5$. But from (1) and (2), $(64)^2 + (45)^2 \neq (65)^2 + (36)^2$. With $a = 6$ and $c = 8$ we have $b = 7$, but $(86)^2 + (67)^2 \neq (87)^2 + (64)^2$. Hence $c^4$ does not end in 6.

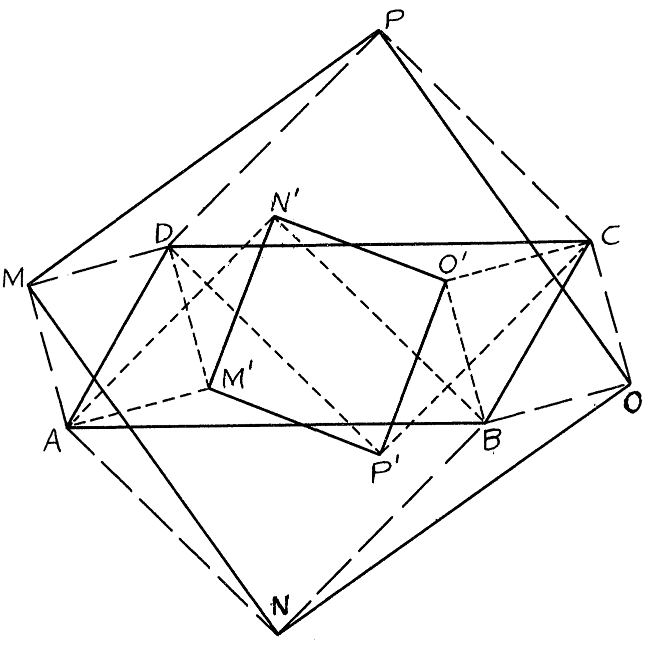
If $c^4$ ends in 1, then $a = 1$, and $c$ might be 3, 7, or 9. But $c^2 < 18$, so $c = 3$. Then $b = 2$, and from (1) and (3), $d = 4$. Thus we have a unique solution.

Also solved by *Leon Bankoff, Los Angeles, Calif; Robert Bonic, University of Chicago;* and *W. B. Carver, Cornell University.*

**Squares Determined by a Parallelogram**

**134.** [May 1952] *Proposed by G. W. Courter, Baton Rouge, Louisiana.*

Using the sides of a parallelogram as hypotenuses, isosceles right triangles are constructed externally (or internally) to the parallelogram. Show that the vertices of the right angles determine a square.




*Solution by A. L. Epstein, Cambridge Research Center, Boston, Mass.* In the figure, the externally constructed isosceles right triangles $PDC$ and $NAB$ are congruent since they have equal hypotenuses, so $PD = AN$. Now $\angle PDM = 2\pi - \pi/2 - (\pi - \angle DAB) = \pi/2 + \angle DAB = \angle MAN$. Furthermore $DM = MA$. It follows that triangles $PDM$ and $MAN$ are congruent, so that $PM = MN$ and $\angle PMD = \angle NMA$. Therefore $\angle PMN = \angle PMD + \angle DMN = \angle NMA + \angle DMN = \angle DMA = \pi/2$. Thus two consecutive sides of the quadrilateral $MNOP$ are equal and perpendicular to each other. By considerations of symmetry any two consecutive sides of $MNOP$ are equal and perpendicular so $MNOP$ is a square.

When the isosceles right triangles are constructed internally on the sides of parallelogram $ABCD$, $P'D = AN'$ and $DM' = M'A$, as before. Also, $\angle P'DM' = \pi - \angle DAB - \pi/2 = \angle DAB - 2(\angle DAB - \pi/4) = \angle M'AN'$. Hence triangles $P'DM'$ and $M'AN'$ are congruent, so that $P'M' = M'N'$ and

$\angle P'M'D = \angle N'M'A$. Now $\angle P'M'N' = \angle P'M'D - \angle N'M'D = \angle N'M'A - \angle N'M'D = \angle DM'A = \pi/2$. Then, by the same argument used in the case of the externally constructed triangles, $M'N'O'P'$ is a square.

Also solved by *Howard Eves, Champlain College; Leon Bankoff, Los Angeles, Calif.; Robert Bonic, University of Chicago; W. B. Carver, Cornell University; R. E. Horton, Lackland AFB, Texas; W. R. Ransom, Tufts College; David Rappaport, Lane Technical High School, Chicago; Charles Salkind, Polytechnic Institute of Brooklyn; and the proposer.*

*Salkind* observed that the diagonals of the squares and of the parallelogram are concurrent. *Eves* remarked that it can be shown that if $ABCD$ is any quadrilateral, then the diagonals $MO$ and $PN$ of $MNOP$ are equal and perpendicular to each other; likewise for the diagonals of $M'N'O'P'$. This latter property is proven synthetically in *School Science and Mathematics*, **47**, 383, (April 1947).

Or, consider the quadrilateral with consecutive vertices $(0,0)$, $(2a,0)$, $(2b,2c)$, $(2d,2e)$. When isosceles right triangles are constructed externally (or internally) on the sides of the quadrilateral as hypotenuses, then the consecutive vertices of the right angles are easily found to be: $(a, \mp a)$, $(a + b \pm c,\ c \pm [a - b])$, $(b + d \pm [e - c],\ c + e \pm [b - d])$, $(d \mp e,\ e \pm d)$, where the top sign applies to the external case, and the bottom sign to the internal case. It follows immediately that the slopes of the diagonals of the derived quadrilaterals are $(\pm a \pm b + c \mp d + e)/(-a + b \mp c + d \pm e)$ and $(\pm a \mp b + c \mp d - e)/(a + b \pm c - d \pm e)$. The product of these slopes is $-1$, so the diagonals are perpendicular. The square of each of the diagonals is $2[a^2 + b^2 + c^2 + d^2 + e^2 + 2(\pm ac - ad \pm be \mp cd)]$, so the diagonals are equal. The midpoints of the diagonals are $\{\tfrac{1}{2}(a + b + d \pm [e - c]),\ \tfrac{1}{2}(\mp[a - b + d] + c + e)\}$ and $\{\tfrac{1}{2}(a + b + d \mp (e - c),\ \tfrac{1}{2}(\pm[a - b + d] + c + e)\}$. It is evident that the midpoint of each diagonal of the external derived quadrilateral coincides with the midpoint of the contrary diagonal of the internal derived quadrilateral. Also, the midpoint of the join of the two points thus determined is the midpoint of each of the lines joining the midpoints of the sides of the original quadrilateral.

Now if $c = e$, and $b = a + d$, then all the aforesaid midpoints coincide at $(a + d,\ c)$. Thus we confirm Salkind's observation and, since the diagonals of the derived quadrilateral are perpendicular and equal, prove the current proposition.

Solutions of this proposition have previously appeared in *Mathematics News Letter*, **4**, 23, (December 1929) and in *National Mathematics Magazine*, **12**, 192, (January 1938). In the latter reference it is shown that the sum of the areas of the external and internal derived squares is equal to that of the squares constructed on two consecutive sides of the parallelogram.

### A Farmer Sells His Eggs

**135.** [May 1952] *Proposed by C. S. Ogilvy, Syracuse University.*

A farmer sells $p/q$ of his eggs plus $p/q$ of an egg to his first customer, $p/q$ of the remaining eggs plus $p/q$ of an egg to his second customer, and so on until all of his eggs have been sold to $n$ customers. Determine necessary and sufficient restrictions on $p$ and $q$ and find the initial number of eggs, if none are to be broken.

*Solution by the Proposer.* We assume that $p/q$ is in lowest terms. The farmer must end up with no eggs. So if $s$ is the number he had just prior to the last sale, then

$$s - (ps/q + p/q) = 0.$$

Hence $p/q = s/(s+1)$, whence $p = q - 1$, the number left before the last sale. Also, before the second-last sale there must have been

$$[q - 1 + (q-1)/q]/[1 - (q-1)/q] \text{ or } q^2 - 1.$$

Furthermore, if before the $k$th sale there were $q^k - 1$, then before the preceding sale [the $(k+1)$th sale] there must have been

$$[q^k - 1 + (q-1)/q]\, q \text{ or } q^{k+1} - 1.$$

This completes the induction, and he must have had $q^n - 1$ eggs for $n$ customers.

Also solved by *Leon Bankoff, Los Angeles, California; W. B. Carver, Cornell University; A. L. Epstein, Cambridge Research Center; M. Morduchow, and Charles Salkind, Polytechnic Institute of Brooklyn.*

### Packing Spheres in a Can

**136.** [May 1952] *Proposed by Corporal P. B. Beilin, Somewhere in Korea.*

What is the maximum number of spheres of radius $r$ which can be placed in a cylindrical can of radius $R$ and height $H$? (Thought of while eating canned pretzel balls.)

*Discussion by W. B. Carver, Cornell University.* There is no possibility that the maximum number of spheres can be expressed as a reasonably simple function of a general $r$, $R$, and $H$. I believe that the very much simpler problem of the maximum number of spheres of diameter 1″ that can be placed in a cubical box 4″ x 4″ x 4″ has not been solved. By putting the spheres in the box in five layers of 16, 9, 16, 9, 16 it is seen that 66 spheres will go in; but I understand that someone found an arrangement by which he could get 67 spheres in the box, but did not prove that 67 was the maximum number. In the absence of a definite solution of this simple problem, the general cylinder problem looks hopeless.

A sphere in space can be touched by just 12 surrounding spheres of the same size, and space can be packed full of spheres in this way, each sphere tangent to 12 others. If at each point of contact of a sphere with the surrounding spheres the tangent plane is drawn, the 12 such planes enclose the sphere in a semi-regular polyhedral box of 12 faces. If the radius of the sphere is the unit of length, each face of the box is a rhombus with edges $\sqrt{6}/2$, the longer diagonal 2, and area $\sqrt{2}$. The volume of this box is then $4\sqrt{2}$, and the ratio of the volume of the sphere to the volume of the box is

$$(4\pi/3)/4\sqrt{2} = \pi/3\sqrt{2} = .7405 \cdots .$$

These boxes are space-filling, that is, they can be packed in space indefinitely with no lost space between them. It follows that when a very large space is filled with closely packed small spheres, each sphere touching 12 surrounding spheres, the spheres will fill approximately 74% of the space.

If then in the cylinder problem the ratios of $R$ and $H$ to $r$ are both very large, the maximum number $N$ of spheres that can be placed in the cylinder will be *approximately*

$$\pi R^2 H/4\sqrt{2} .$$

To state this more accurately,

$$4\sqrt{2}N/\pi R^2 H \to 1 \text{ when } R \to \infty \text{ and } H \to \infty.$$

**A Prime Number Property**

**137.** [May 1952] *Proposed by W. T. Cleagh, Jacksonville, Florida.*

Let $N = \left| \Pi p_i^{\alpha_k} \pm \Pi p_j^{\alpha_m} \right|$ where the sets $p_i$ and $p_j$ together constitute the first $n$ primes and the $\alpha_k$ and $\alpha_m$ are arbitrary positive integers. Show that $N$ is a prime if $N$ is less than the square of the $(n + 1)$th prime. For example: $(2)^2(3)(5)(7)(11) - (13)(17)(19) = 421 < (23)^2$, so 421 is a prime.

*Solution by J. S. Shipman, Laboratory for Electronics, Inc., Boston, Massachusetts.* To find the divisors of $N$, it is only necessary to divide by all primes $\leqq \sqrt{N}$ and examine the remainder. By hypothesis, $\sqrt{N}$ is less than the $(n + 1)$th prime, so the $(n + 1)$th prime is not a divisor of $N$. None of the first $n$ primes divide $N$, for the $p_i$ and $p_j$ are disjoint and the $\alpha_k$ and $\alpha_m$ are strictly positive, from which it follows that dividing by a $p_i$, say, leaves a remainder of $\pm \prod_{p_j} \alpha_m$.
Since none of the primes $\leqq \sqrt{N}$ divides $N$, it follows that $N$ is prime.

Also solved by *Samuel Holland, Jr., University of Chicago.*

## An Interesting Identity

**138.** [May 1952] *Proposed by D. Arany, Budapest, Hungary.*

Establish the following identity:
$[(AP)^2 - (AH)^2]\tan A + [(BP)^2 - (BH)^2]\tan B + [(CP)^2 - (CH)^2]\tan C = (PH)^2(\tan A + \tan B + \tan C)$, where $P$ is an arbitrarily chosen point in the plane of the triangle $ABC$ and $H$ is the orthocenter of $ABC$.

**I.** *Solution by Howard Eves, Champlain College.* It is well known (cf. M'Clelland, *The Geometry of the Circle*, p. 99) that

$$\alpha(AP)^2 + \beta(BP)^2 + \gamma(CP)^2 = \alpha(AH)^2 + \beta(BH)^2 + \gamma(CH)^2 + (\alpha + \beta + \gamma)(PH)^2,$$

where $\alpha$, $\beta$, $\gamma$ are the areal coordinates of $H$ for triangle $ABC$. Since the areal coordinates of $H$ are proportional to $\tan A$, $\tan B$, $\tan C$ the desired identity follows.

**II.** *Solution by Leon Bankoff, Los Angeles, Calif.* In similar triangles $AHE$ and $BEC$, $BC/AH = BE/AE = \tan A$. In similar triangles $BHF$ and $AFC$, $AC/BH = CF/BF = \tan B$. In similar triangles $CHE$ and $ABE$, $AB/CH = BE/EC = \tan C$. It follows that $BC = AH \tan A$, $AC = BH \tan B$, and $AB = CH \tan C$.

Now by the cosine law:

$$[(PH)^2 + (AH)^2 - (AP)^2] = 2(PH)(AH) \cos PHA$$

$$[(PH)^2 + (BH)^2 - (BP)^2] = 2(PH)(BH) \cos PHB$$

$$[(PH)^2 + (CH)^2 - (CP)^2] = 2(PH)(CH) \cos PHC$$

Multiplying by the tangents and then substituting the values established above, we have:

$$\tan A[(PH)^2 + (AH)^2 - (AP)^2] = 2(PH)(AH)\tan A \cos PHA = 2(PH)(BC) \cos PHA$$

$$\tan B[(PH)^2 + (BH)^2 - (BP)^2] = 2(PH)(BH)\tan B \cos PHB = 2(PH)(AC) \cos PHB$$

$$\tan C[(PH)^2 + (CH)^2 - (CP)^2] = 2(PH)(CH)\tan C \cos PHC = 2(PH)(AB) \cos PHC$$

We now show that the sum of the right members of these three expressions vanishes, whereupon the sum of the left members equals zero and the announced identity is established.

$$BC \cos PHA + AC \cos PHB + AB \cos PHC$$

$$= BC \cos PHA - AC \cos (PHA + AHE) - AB \cos (FHA - PHA)$$

$$= BC \cos PHA - AC (\cos PHA \cos AHE - \sin PHA \sin AHE)$$

$$- AB(\cos FHA \cos PHA + \sin FHA \sin PHA)$$

$$= \cos PHA(BC - AC \cos AHE - AB \cos FHA)$$

$+ \sin PHA(AC \sin AHE - AB \sin FHA)$

$= \cos PHA(BC - AC \cos C - AB \cos B) + \sin PHA(AC \sin C - AB \sin B)$

$= \cos PHA(BC - DC - BD) + \sin PHA(AD - AD) = 0.$

**III.** *Solution by W. B. Carver, Cornell University.* If we take the circumcircle of the triangle as the unit circle in the complex plane, vertices $A$, $B$, $C$, will correspond to three "turns", $t_i$, $i = 1, 2, 3$; a turn being a complex number of absolute value 1, so that $t_i \bar{t}_i = 1$. The orthocenter $H$ corresponds to the number $t_1 + t_2 + t_3$, and we let $P$ correspond to the number $z$.

Then the square of the distance from the vertex $t_i$ to $P$ is

$$(z - t_i)(\bar{z}_i - \bar{t}_i),$$

the square of the distance from the vertex $t_i$ to $H$ is

$$(t_j + t_k)(\bar{t}_j + \bar{t}_k),$$

the square of the distance $PH$ is

$$(z - t_1 - t_2 - t_3)(\bar{z} - \bar{t}_1 - \bar{t}_2 - \bar{t}_3),$$

and the tangent of the angle at the vertex $t_i$ is

$$(t_j - t_k)\sqrt{-1}/(t_j + t_k).$$

When these expressions are substituted in the relation to be proved, one finds that it reduces readily to an identity in $z$, $\bar{z}$, $t_1$, $t_2$, and $t_3$.

Also solved by *A. L. Epstein, Cambridge Research Center, Massachusetts,* using analytic geometry; and *Arthur Gregory, Albuquerque, New Mexico.*

## TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Trickies will be alternated with Quickies in future issues if our readers evidence enough interest. Send in your favorite Trickies.

**T 1.** If you were a taxicab driver and a fair female fare had four pieces of luggage, one weighing 17 lbs., another weighing 73 lbs., a third 41 lbs., and the fourth 29 lbs., what did the taxicab driver weigh? [*Submitted by Clay Perry.*]

**T 2.** Prove that if the top 26 cards of an ordinary shuffled deck contain more red cards than there are black cards in the bottom 26, then there are in the deck at least three consecutive cards of the same color. [*Submitted by Leo Moser.*]

**T 3.** Let $F$ be a field in which every element is a square. Prove that it is also true that every element is a square root. [*Submitted by J. L. Brenner.*]

**T 4.** Three islands in the South Pacific are located at the vertices of an equilateral triangle. An airplane making the "grand tour" on a perfectly calm day flew the first two legs in eighty minutes each, and the third leg in an hour and twenty minutes. Why did it take so long to fly the last leg? [*Submitted by J. M. Howell.*]

**T 5.** A rich person who possessed a very expensive Swiss watch once bragged to a poor friend that not only was his watch an automatic winding one, but it lost only $1\frac{1}{4}$ sec. per day. The friend remarked that the watch would indicate the correct time only about once in a century. This annoyed the rich man who demanded to know of a better watch. The friend said that his four-year old daughter had just gotten a watch which though inexpensive at least did indicate the correct time twice every day. How accurate was the daughter's watch? [*Submitted by M. S. Klamkin.*]

## SOLUTIONS

**S 1.** This problem is more effective orally than when printed, since the entire thing depends upon the opening phrase. If *you* were the taxicab driver, his weight is your weight.

**S 2.** A little reflection, or algebra, reveals that the number of red cards in the top 26 must always equal the number of black cards in the bottom 26. Hence, by the rules of logic, the whole assertion following the word "that" is correct, no matter what follows the word "then".

**S 3.** In any multiplicative system, every number is the square root of its own square.

**S 4.** It took no longer than on the other two legs. 80 minutes = 1 hour 20 minutes.

**S 5.** The cheap watch must have gained or lost 24 hours per day. It was a stationary toy watch.

## QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

"**Q** 51. [January 1952] Compute the first period of the repeating decimal equivalent to $1/7^2$." *M. S. Klamkin* calls attention to a method due to J. W. L. Glaisher [see, L. E. Dickson, *History of the Theory of Numbers*, Vol. I, p. 170]. To get the period, start with 1 and divide repeatedly by 5, listing the quotient and securing the next divident by prefacing the last quotient by its remainder. Thus,

| *Division* | *Quotient* | *Remainder* | *Partial Decimal Equivalent* |
|---|---|---|---|
| 1/5 | 0 | 1 | 0.0 |
| 10/5 | 2 | 0 | 0.02 |
| 02/5 | 0 | 2 | 0.020 |
| 20/5 | 4 | 0 | 0.0204 |
| 04/5 | 0 | 4 | 0.02040 |
| 40/5 | 8 | 0 | 0.020408 |

The process is repeated until the digits repeat.

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**Q 75.** If a particle traversed one foot in one second starting and ending at rest, show that at some point the acceleration was $\geqq 4$ ft./sec.$^2$ It is assumed that $v$ and $a$ are continuous functions of $t$. [*Submitted by M. S. Klamkin.*]

**Q 76.** Of all the surfaces of revolution which connect two given generated circles, and which have a meridian of a given length, a catenoid will have the least area. [*Submitted by J. H. Butchart.*]

**Q 77.** Divide a number $n$ into two parts so that their product is a maximum. [*Aaron Bakst* in *The Mathematics Teacher*, **45**, 115, Feb., 1952.]

**Q 78.** $n$ boys live in different houses on a straight street. Where should they meet in order that the sum of the distances walked be a minimum? [*Submitted by J. M. Howell.*]

**Q 79.** If $\phi(a)^{\phi(a)} \equiv a$, solve $x^{1/kx} = \phi(a)$. [*Submitted by M. S. Klamkin.*]

**Q 80.** In assembling a jig saw puzzle, let us call the fitting together of two pieces a "move", independently of whether the pieces consist of single pieces or of blocks of pieces already assembled. What procedure will minimize the number of moves required to solve an $n$-piece puzzle? What is this minimum number? [*Submitted by Leo Moser.*]

## ANSWERS

**A 75.** If we plot $v$ against $t$, the area under the curve must be the same (1 square unit) as that of an isosceles triangle having the same base and an altitude of 2. The slopes of the sides of the triangle are ±4. Part of the $v$, $t$ curve must fall outside the triangle or coincide with its sides. Thus the slope, $a$, of the curve is numerically $\geqq 4$ at some point.

**A 76.** The center of gravity of the meridian is closest to the axis of the surface when the meridian is a catenary. Hence the catenoid will have the least area by Pappus' theorem. (The area generated by revolving a plane curve about an axis in its plane is equal to the length of the curve multiplied by the circumference of the circle described by its center of gravity.)

**A 77.** Let one part be $x$, then $(n - x)x = M$, so $x = [n \pm \sqrt{n^2 - 4M}]/2$. Since $x$ is real, the maximum value of $M$ is such that $n^2 = 4M$. Hence, $x = n/2 = (n - x)$.

**A 78.** It is well-known to statisticians that the sum of the absolute values of deviations about an arbitrary point is a minimum about the median. Therefore, if there is an odd number of houses, the boys should meet at the center one, or halfway between the two center ones if the number of houses is even. In the case of $n$ boys living in $m$ houses, $m \leqq n$, the meeting place should be at the median of the weighted distances.

**A 79.** Let $\frac{1}{x} = y$, then we have $y^y = [\phi(a)]^{-k} = \{\phi[\phi(a)]^{-k}\}^{\{\phi[\phi(a)]^{-k}\}}$

Hence $x = \{\phi[\phi(a)]^{-k}\}^{-1}$.

**A 80.** Since the number of pieces is originally $n$ and finally 1, and since every move reduces the number of pieces by 1, the total number of moves required is $n - 1$, and is clearly independent of the procedure. (So long, of course, as we never separate pieces already joined.)

# CURRENT PAPERS AND BOOKS

*Edited by*

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

*The Nature of Number.* By Roy Dubisch, The Ronald Press Company, New York, 1952, IX + 159 pages, $4.00.

Like the books by E. T. Bell and Edna E. Kramer, which are reviewed in this issue, one objective of the book now under consideration is to interest the layman as well as the student of mathematics in some of the basic ideas of modern mathematics. However, Professor Dubisch achieves this by "portraying the development of a single line of mathematical thought from its most primitive beginnings to contemporary times." A second objective is to reveal the abstract nature of elementary mathematics.

In the first two chapters a brief summary of the development of the concept of whole number is given. After this historical approach a set of assumptions pertaining to the undefined term, positive integer, is considered. Negative integers and fractions are treated similarly. Then irrational numbers, complex numbers, quaternions and, finally, various algebras are discussed.

Reviews and problems for which the answers are given appear frequently. In an appendix a more rigorous treatment is presented for the student of mathematics.

The clarity and simplicity of the author's exposition should appeal to the layman. This book should be helpful to teachers of arithmetic and algebra and to those who are beginning their study of abstract mathematics.

Edith R. Schneckenburger

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*Mathematics, Queen and Servant of Science.* By E. T. Bell, McGraw-Hill Book Company, New York, 1951, XX + 437 pages, $5.00.

This book is a revision and integration of two of the author's earlier volumes: *The Queen of the Sciences*, The Williams and Wilkins

Company, 1931, and *The Handmaiden of the Sciences*, the Williams and Wilkins Company, 1937. Readers of these popular books will recall that the first book was written for the Century of Progress Series to describe the development of mathematics during the last century. In the second book some of the applications of mathematics in science were considered. In the new volume the material of the two earlier books has been amplified and woven together in a very skillful fashion.

Algebra, analysis, arithmetic and geometry receive equal consideration. Various kinds of numbers, algebras, calculuses, geometries, and the relations between them are discussed. In presenting new developments in each field the power of generalization and abstraction is stressed.

No attempt is made to give the complete history of the growth of each new concept. However, the many historical facts which are included add greatly to the interest of the material. Also, they reveal the importance of looking at old facts in a new way.

Applications are drawn chiefly from astronomy and physics. In discussing these applications the author emphasizes the influence of science on the development of mathematics as well as the usefulness of mathematics to the scientist.

As in his previous books the author's informal style creates the impression that he is conversing with the reader. One appreciates the occasional touches of humor.

This book can be read with understanding by the layman for whom it is intended. No doubt it can be read with greater appreciation by students of mathematics. It should be recommended especially to teachers of mathematics or physical science and to undergraduate majors who are planning to become teachers or research workers in these fields. Cross-references which are given easily by the use of section numbers, a good index, and a separate listing of the names appearing in the book increase its usefulness as a reference book.

Edith R. Schneckenburger

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*The Main Stream of Mathematics*. By Edna E. Kramer, Oxford University Press, New York, 1951, XII + 321 pages.

This book presents an account of the development of some of the fundamental concepts of modern mathematics. The story begins with the Hindu number system and ends with Cantor's theory of the infinite. However, each of the thirteen chapters of this book is a unit in itself and contains a wealth of historical material as well as applications of current interest pertaining to the principal theme of the chapter.

In Chapter I the Hindu - Arabic number system is described. In succeeding chapters the following topics receive major emphasis:

geometric forms; the basic principles and symbolism of algebra; irrational numbers; equations; trigonometric functions; analytic geometry; the theory of probability and its applications in statistics; series; differential and integral calculus; systems of postulates; relativity theories; the infinite.

In order to describe more adequately the scope of this book and to convey some impression of the author's technique a more detailed account of one chapter is given. Chapter 4 is entitled "The Mathematics of the Forbidden." The introductory paragraphs present a brief account of Pythagoras and the Pythagoreans. Then the Pythagorean theorem and the discovery of incommensurable numbers are considered. The effect of modern algebraic symbolism upon the use of incommensurable numbers is shown. Then, the following topics are discussed briefly: repeating and non-repeating decimals; the square root spiral; cube roots; use of incommensurable ratios in classic art; exponential notation for large and small numbers; laws of exponents illustrated by applications in physics; logarithms, including the irrational Napierian base.

One is impressed by the skillful interweaving of the old and the new and by the effective use of applications both in introducing topics and in providing meaning. Numerous diagrams and illustrations are given. A list of references is included.

This book was written primarily for non-mathematicians and does not require much technical background. However, it will be of interest to many students of mathematics. Teachers of mathematics on both the secondary and collegiate level will find in this book interesting stories and illustrations that can be used in the classroom.

Edith R. Schneckenburger

*Measure Theory*. By Paul R. Halmos. D. Van Nostrand Company, New York, 1950 xi + 304 pages $5.90.

This second volume in the *University Series in Higher Mathematics* is intended for use as a textbook in graduate courses in Measure Theory. It should also prove useful for reference purposes with its collection of theorems, its well chosen problems, and its penetrating comments on the subjects which it touches. The necessary prerequisites in algebra and analysis are given in an introductory section which lists the knowledge required in the various chapters. The paragraphs on Topological Spaces and Topological Groups constitute a brief (5 page) summary of the principal concepts of these subjects.

The scope of the book may be judged by a partial listing of its chapter headings: Sets and Classes, Measures and Outer Measures, Measureable Functions, Integration, General Set Functions, Probability, Haar Measure, Measure and Topology in Groups. There is included a list of References, a Bibliography, a useful List of Symbols, and an adequate

Index.

Unlike some textbooks, the exercises in Halmos' book are not trivial. They contain well chosen special cases to illustrate the theory, alternative methods of proof of theorems, and additional definitions and theorems to extend the material of the text. To a reader with sufficient mathematical background, the problems are a most interesting feature of the book. Another good feature is the use of a bar to signalize the end of a proof, thus separating the completion of the proof of one theorem from comments leading to the next theorem.

This book is to be recommended for the purposes for which it was written. Its publication should have a beneficial influence upon graduate instruction in Mathematics.

Harry M. Gehman

*The Design and Analysis of Experiments*. By Oscar Kempthorne. Wiley, New York, 1952. $8.50.

The *Wiley Publications in Statistics* form an exceptionally fine and interesting series. Those in the subseries *Mathematical Statistics* by Wald, Feller, Dwyer and Kempthorne have been of particular interest to the reviewer as providing a mathematical background for understanding the practice. As he modestly indicates in his introduction Professor Kempthorne is especially well qualified to write this book. He worked under Yates for a number of years and has made numerous significant contributions of his own.

The excellence of organization of the material is evident from the table of contents. The book is extremely detailed and a greater part of it can be scanned rapidly if one is trying to get a general idea of where the details are to be found. The first several chapters relate the statistical design of experiments to the scientific method. The next several chapters give a systematic treatment of general linear hypothesis theory. The remainder of the first twelve foundation chapters discuss randomization of uncontrolled parameters. This introductory material covers two hundred odd pages. The remaining four hundred pages give a detailed and systematic treatment of practical design.

As an applied mathematician interested in a reference book in this field, the reviewer looked at the book for scanability, clarity and completeness. It was possible to scan it rapidly and obtain a clear picture of the field. Sample discussions were read and were clear enough not to require rereading. The book appears to be complete and in addition has excellent references to the literature for further information.

N. G. Parke

Dear Professor James:

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